Department of Commerce

University of Calcutta

Study Material

Cum

Lecture Notes

Only for the Students of M.Com. (Semester II)-2020

University of Calcutta

(Internal Circulation)

Dear Students,

Hope you, your parents and other family members are safe and secured. We are going through a world-wide crisis that seriously affects not only the normal life and economy but also the teaching-learning process of our University and our department is not an exception.

As the lock-down is continuing and it is not possible to reach you face to face class room teaching. Keeping in mind the present situation, our esteemed teachers are trying their level best to reach you through providing study material cum lecture notes of different subjects. This material is not an exhaustive one though it is an indicative so that you can understand different topics of different subjects. We believe that it is not the alternative of direct teaching learning.

It is a gentle request you to circulate this material only to your friends those who are studying in Semester II (2020).

Stay safe and stay home.

Best wishes.

Paper: CC203:

Operations Research (OR)

Module I: Linear Programming Problem

The topic we have already covered:

1. Basic idea of Linear Programming Problem (LPP)
2. Formulation of LPP- (a) Maximisation and (b) Minimisation Type Problems
3. **Solution of LPP:**
   - ***Graphical Method:***
     (a) Solution of Maximisation Type LPP through Graphical Method
     (b) Solution of Minimisation Type LPP through Graphical Method

❑ Simplex Method

In class lectures I have discussed that it is not possible to obtain graphical solution to the LPP of more than two variables. On graph paper, only two variables can be accommodated. The analytic solution is also not possible because the tools of analysis are not well studied to handle inequalities. The most commonly used method for finding out the optimal solution to LPP is the simplex method which was developed by G. Dantzig in 1947.

The simplex method is a computational procedure i.e. an algorithm for solving linear programming problems. It is an iterative technique of optimisation. The simplex method consists of:

(i) Finding a trial basic feasible solution (extreme point) to the constraint equations.
(ii) Testing whether the initial basic feasible solution (IBFS) is optimal or not.
(iii) Improving, if required, the first trial solution by set of rules and repeating the process until we reach an optimal solution.

➢ The Simplex Method for Maximisation Problems:

Question No. 1:

Solve the following using Simplex Method:

Maximise $Z = 8x_1 + 16x_2$

Subject to,

$x_1 + x_2 \leq 200$

$x_2 \leq 125$

$3x_1 + 6x_2 \leq 900$

x_1 and x_2, ≥ 0

Solution:

Introducing necessary slack variables, the given LPP becomes:

Maximise $Z = 8x_1 + 16x_2 + 0S_1 + 0S_2 + 0S_3$

Subject to,

$x_1 + x_2 + S_1 = 200$

$x_2 + S_2 = 125$

$3x_1 + 6x_2 + S_3 = 900$

x_1, x_2, S_1, S_2 and $S_3 \geq 0$

SIMPLEX TABLEAU- I (i.e. INITIAL SIMPLEX TABLEAU)

C_j (Contribution per unit)			8	16	0	0	0	Minimum Ratio or Replacement Ratio
Basic Variable Coefficient (C_B)	**Basic Variables (B)**	**Basic Variable Value b (=X_B)**	x_1	x_2	S_1	S_2	S_3	
0	S_1	200	1	1	1	0	0	200/ 1 = 200
0	S_2	125	0	1*	0	1	0	125/ 1 = 125
0	S_3	900	3	6	0	0	1	900/ 6 = 150
		Z_j	0	0	0	0	0	
Net Evaluation Row or Net Contribution Per Unit i.e. $\Delta_j = (C_j - Z_j)$			8	16 ↑	0	0	0	

Since all the values of $C_j - Z_j$ row are not either zero or negative, the above solution is not optimal. In order to obtain optimal solution we need to improve the above till all the values of $C_j - Z_j$ row are either zero or negative (This is the rule for having optimal solution in case of a maximisation type of LPP). Since it is a maximisation type LPP, the $C_j - Z_j$ row element having the ***maximum value*** shall be considered to find out key column. Here 16 in NER is the maximum value. Now each element of basic variable value is divided by corresponding element in key column in order to find out minimum ratio. The minimum ratio is the minimum value among the all elements. Here it is determined as 125. The row corresponding to 125 i.e. minimum ratio is termed as key row or pivot row. The element falling in the intersection of key row and key column is called the key/ pivot element. Here it is "1". In the next tables, we will find out the optimal solution.

(a) **New Key Row Element**

$$= \frac{\text{Old Key Row Element}}{\text{Key Element}}$$

(In Simplex Tableau I, the key element is 1 i.e. intersection value of Key Row and Key Column). The Sky Blue colour column is Key Column and Saffron Colour row row is Key Row.Row.

(b) **Other than Key Row Element** =

Old Row Element (–) Corresponding Key Row Element × $\frac{\text{Correspondinding Key Cloumn Value}}{\text{Key Element}}$

= Old Row Element (–) Corresponding Key Row Element × Fixed Ratio

SIMPLEX TABLEAU- II

		C_j (Contribution per unit)	8	16	0	0	0	
Basic Variable Coefficient (C_B)	Basic Variables (B)	Basic Variable Value b (=X_B)	x_1	x_2	S_1	S_2	S_3	Minimum Ratio or Replacement Ratio
0	S_1	75	1	0	1	- 1	0	75/ 1 = 75
16	x_2	125	0	1	0	1	0	125/ 0 = -
0	S_3	150	3*	0	0	- 6	1	150/ 3 = 50
		Z_j	0	16	0	16	0	
		$\Delta_j = (C_j – Z_j)$	8 ↑	0	0	- 16	0	

Since all the values of $C_j – Z_j$ row are neither zero nor negative, the above solution is also not an optimal solution. We need another iteration to find out optimal solution, which is shown in next table.

SIMPLEX TABLEAU- III

		C_j (Contribution per unit)	8	16	0	0	0	
Basic Variable Coefficient (C_B)	Basic Variables (B)	Basic Variable Value b (=X_B)	x_1	x_2	S_1	S_2	S_3	Minimum Ratio or Replacement Ratio
0	S_1	25	0	0	1	1	- 1/3	
16	x_2	125	0	1	0	1	0	
8	x_1	50	1	0	0	- 2	1/3	
		Z_j	8	16	0	0	8/3	
		$\Delta_j = (C_j – Z_j)$	0	0	0	0	- 8/3	

Since all the values of $C_j – Z_j$ row are either zero or negative, so the above solution is optimal. Therefore, the optimal solution is $x_1 = 50$, $x_2 = 125$ and Max. $Z = 8 \times 50 + 16 \times 125 = 2400$.

➢ Artificial Variable Techniques:

In last LPP, we observed constraints with less than or equal to (i.e. ≤) type. This property together with the fact that the right hand side (R.H.S) of all the constraints is non-negative, provide us with a ready starting initial basic feasible solution (IBFS) that comprises of all slack variables.

But in many LPP, only slack variables cannot provide such a solution, where the left hand side (L.H.S) of all constraints is of either "≥" or "=" type. In such a case, we introduce non-negative artificial variables to the left hand side. The purpose of introducing artificial variables is just to obtain an initial basic feasible solution (IBFS). However, since such artificial variables have ***no physical meaning*** in the original model (hence the variables are called artificial variables), provisions must be made to make zero level at the optimum iteration. In other words, we use them to start the solution and abandon them once their work has been over. There are two methods for removing artificial variables from the solution:

(a) Big 'M' Method or Method of Penalty due to A. Charnes
(b) The Two Phases Simplex Method due to Dantzig, Orden and Wolfe.

Here, we will restrict our discussion to only Big M Method.

Big M Method:

It has already been discussed since artificial variables do not represent any quantity relating to the decision problem, they must be driven out of the system and must not show in the final or optimal solution. This can be done by assigning an extremely high cost to them. Generally a value 'M' is assigned to each artificial variable, where M represents a number higher than any finite number. That is why the method of solving problems where artificial variables are involved are termed as the Big-M Method.

Thus, when the LPP is of minimisation type, we assign in the objective function a coefficient of + M to each of the artificial variables. On the other hands LPP with objective function of maximisation type, each artificial variable introduced has a coefficient – M.

➢ The Simplex Method for Minimisation Problems:

Question No. 2:

Solve the following using Simplex Method:

Minimise $Z = 2x_1 + 8x_2$

Subject to,

$5x_1 + 10x_2 = 150$

$x_1 \leq 20$

$x_2 \geq 14$ and $x_1 \geq 0$

Solution:

According to the above constraints, the variable x_2 will have minimum value = 14. Therefore, let us assume that $x_2 = 14 + x_2'$. Hence the given LPP can be re-written as:

Minimise $Z = 2x_1 + 8(14 + x_2') = 2x_1 + 8x_2' + 112$

Subject to,

$5x_1 + 10(14 + x_2') = 150$

or, $5x_1 + 10x_2' = 10$

$x_1 \leq 20$

x_1 and $x_2' \geq 0$

Introducing necessary slack variable S_1 and artificial variable A_1, the given LPP becomes:

Minimise $Z = 2x_1 + 8x_2' + 0S_1 + MA_1 + 112$

Subject to,

$5x_1 + 10x_2' + A_1 = 10$

$x_1 + S_1 = 20$

x_1, x_2', S_1 and $A_1 \geq 0$

SIMPLEX TABLEAU- I (i.e. INITIAL SIMPLEX TABLEAU)

C_j (Contribution per unit)			2	8	0	M	Minimum Ratio or Replacement Ratio
Basic Variable Coefficient (C_B)	Basic Variables (B)	Basic Variable Value b (=X_B)	x_1	x_2'	S_1	A_1	
M	A_1	10	5	10*	0	1	10/10 = 1
0	S_1	20	1	0	1	0	20/0 = -
		Z_j	5M	10M	0	M	
Net Evaluation Row (NER) or $\Delta_j = (C_j - Z_j)$			2– 5M	8 – 10M ↑	0	0	

The rule for optimisation in case of minimisation problem is that all values of NER i.e $C_j - Z_j$ are either zero or **positive**. But in the above solution, two values in NER are negative (i.e. 2– 5M and 8 – 10M). Therefore, the above solution is not an optimal solution. We need to further improve the initial basic feasible solution (IBFS) in the next iterations.

Out of these two negative values, the minimum value is 8 - 10M. The column containing 8 – 10M value is referred to as Key Column and marked by upper arrow.

SIMPLEX TABLEAU- II

C_j (Contribution per unit)			2	8	0	M	Minimum Ratio or Replacement Ratio
Basic Variable Coefficient (C_B)	Basic Variables (B)	Basic Variable Value b (=X_B)	x_1	x_2'	S_1	A_1	
8	x_2'	1	½*	1	0		1÷ 1/2 = 2
0	S_1	20	1	0	1		20/1 = 20
		Z_j	4	8	0		
		$\Delta_j = (C_j - Z_j)$	- 2 ↑	0	0		

The above solution is also not optimal since one $C_j - Z_j$ row contains negative value. Therefore, the solution needs further improvement with the help of following table:

SIMPLEX TABLEAU- III

2	x_1	2	1	2	0		
0	S_1	18	0	- 2	1		
		Z_j	2	4	0		
		$\Delta_j = (C_j - Z_j)$	0	4	0		

The all the values of $C_j - Z_j$ row are either zero or positive, so the above solution is optimal. Therefore, the optimal solution is $x_1 = 2$, $x_2' = 0$ and Min. Z = 2 × 2 + 8 × 0 + 112 = 116.

CC203 OR: Operations Research

(Prof. J.K. Das)

Module –I

Unit3: Simulation Models
Unit 4: Decision Theory and Game Theory

SIMULATION

19.1 INTRODUCTION

In previous chapters we have discussed a number of analytical methods (or procedures) to provide an optimal solution to a given problem. However, there are certain real world problems which although mathematical in nature involve variables whose values are determined by chance. Thus solution to problems in such cases is obtained in terms of expected (maximum or minimum) pay-off value.

Simulation is a numerical solution method that seeks optimal alternatives (strategies) through a trial and error process. The simulation approach can be used to study almost any problem that involves uncertainty, i.e. problems where probability distribution of variables is known in advance or specified can be solved by this technique. However, simulation approach requires an analogous physical model to represent mathematical and logical relationship among variables of the problem under study. After having constructed the desired model, the simulation approach evaluates each alternative (measure of performance) by generating a series of values of random variables on paper over a period of time within the given set of conditions or criteria. This process of generating series of values one after another to understand the behaviour of the system (operational informations) is called *executing (running or experimenting)* model on computers.

It is the availability of computers which makes it possible to deal with an extraordinarily large quantity of details which can be incorporated into a model and the ability to manipulate the model over many experiments (i.e. replicating all the possibilities that may be imbedded in the external world and events would seem to recur). The use of the word simulation can be traced to the mathematicians Von Neumann and Ulam in the late 1940s when they developed the term *Monte Carlo analysis* while trying first to break the casino at Monte Carlo and subsequently, applying it to solution of nuclear shielding problems that were either too expensive for physical experimentation or too complicated for treatment by known mathematical technique.

An agreed definition for the world simulation has not been reached so far, however few definitions are stated as:

- *A simulation of a system or an organism is the operation of a model or simulator which is a representation of the system or organism. The model is amenable to manipulation which would be impossible, too expensive or unpractical to perform on the entity it portrays. The operation of the model can be studied and for it, properties concerning the behaviour of the actual system can be inferred.*

— Shubik

This definition is broad enough to be applied equally to military war games, business games, economic models, etc. In this view simulation involves logical and mathematical constructs that can be manipulated on a digital computer using iterations or successive trials.

- *Simulation is the process of designing a model of a real system and conducting experiments with this model for the purpose of understanding the behaviour (within the limits imposed by a criterion or set of criteria) for the operation of the system.*

— Shannon

- *Simulation is a numerical technique for conducting experiments on a digital computer, which involves certain types of mathematical and logical relationships necessary to describe the behaviour and structure of a complex real-world system over extended periods of time.*

— Naylor *et. al.*

Few other definitions of simulation are as under:

- *"X simulated Y" is true if and only if*
  (i) *X and Y are formal systems,*
  (ii) *Y is taken to be the real system,*
  (iii) *X is taken to be an approximation to the real system, and*
  (iv) *The rules of validity in X are non-error-free, otherwise X will become the real system.*
- *Simulation is the use of a system model that has the designed characteristics of reality in order to produce the essence of actual operation.* — Churchman

For operations research, simulation is a problem solving technique which uses a computer-aided experimental approach to study problems that cannot be analysed using direct and formal analytical methods. As a result simulation can be thought of as a last resort technique. It is not a technique which should be applied in all cases. However, Table 19.1 highlights what simulation is and what it is not.

Table 19.1 Simulation what it is/not

It is	*It is not*
• a technique which uses computers.	• an analytical technique which provides exact solution.
• an approach for reproducing the processes by which events of chance and change are created in a computer.	• a programming language but it could be programmed into a set of commands which can form a language to facilitate the programming of simulation.
• a procedure for testing and experimenting on models to answer what if........, then so and so......types of questions.	

Simulation is a fast and relatively inexpensive method of performing 'experiments' on the computer. For example,

1. In inventory control, the problem of determining the optimal replenishment policy arises due to the probabilistic (stochastic) nature of demand and lead time. Thus, instead of manually trying out the three replenishment alternatives for each level of demand and lead time for a period of one year and then selecting the best one, we process data (called *experiment*) on the computer and obtain the results in a very short time at a very small cost.

2. In queuing theory, the problem of balancing the cost of waiting against the cost of idle time of service facilities in the system arises due to the probabilistic nature of the inter-arrival times of customers and the time taken to complete service to the customer. Thus, instead of trying out in actual manually with data to design a single server queuing system, we process the data on computers and obtain the expected value of various characteristics of the queuing system such as idle time of servers, average waiting time, queue length, etc.

Unlike various analytical methods there are no written and fixed rules to guide the formulation of simulation models. Each application of simulation is different from the other and ad hoc to a large extent.

19.2 STEPS OF SIMULATION PROCESS

The process of simulating a system consists of following steps:

Step 1 Identify the problem

If any inventory system is being simulated, then the problem may concern the determination of the size of order (number of units to be ordered) when inventory level falls up to the reorder level (point).

Step 2 (a) Identify the decision variables

(b) Decide the performance criterion (objective) and decision rules

In the context of the above defined inventory problem, the demand (consumption rate), lead time and safety stock are identified as decision variables. These variables shall be responsible to measure the performance of the system in terms of total inventory cost under the decision rule—when to order.

Step 3 Construct a numerical model

A numerical model is constructed to be analysed on the computer. Sometimes the model is written in a particúlar simulation language which is suited for the problem under analysis.

Step 4 Validate the model

Validation of the model is necessary to ensure whether it is truly representing the system being analysed and the results will be reliable.

Step 5 Design the experiments

Conduct experiments with the simulation model by listing specific values of variables to be tested (i.e. list courses of action for testing) at each trial (run).

Step 6 Run the simulation model

Run the model on the computer to get the results in the form of operating characteristics.

Step 7 Examine the results

Examine the results of problem as well as their reliability and correctness. If the simulation process is complete, then select the best course of action (or alternative) otherwise make desired changes in model decision variables, parameters or design, and return to Step 3.

The steps of simulation process are also shown in Fig. 19.1.

19.3 ADVANTAGES AND DISADVANTAGES OF SIMULATION

Advantages

1. This approach is suitable to analyse large and complex real-life problems which cannot be solved by usual quantitative methods.
2. Simulation allows the decision-maker to study the interactive system variables and the effect of changes in these variables on the system performance in order to determine the desired one.
3. Simulation experiments are done with the model, not on the system itself. It also allows to include additional information during analysis that most quantitative models do not permit. In other words, simulation can be used to experiment on a model of a real situation without incurring the costs of operating on the system.

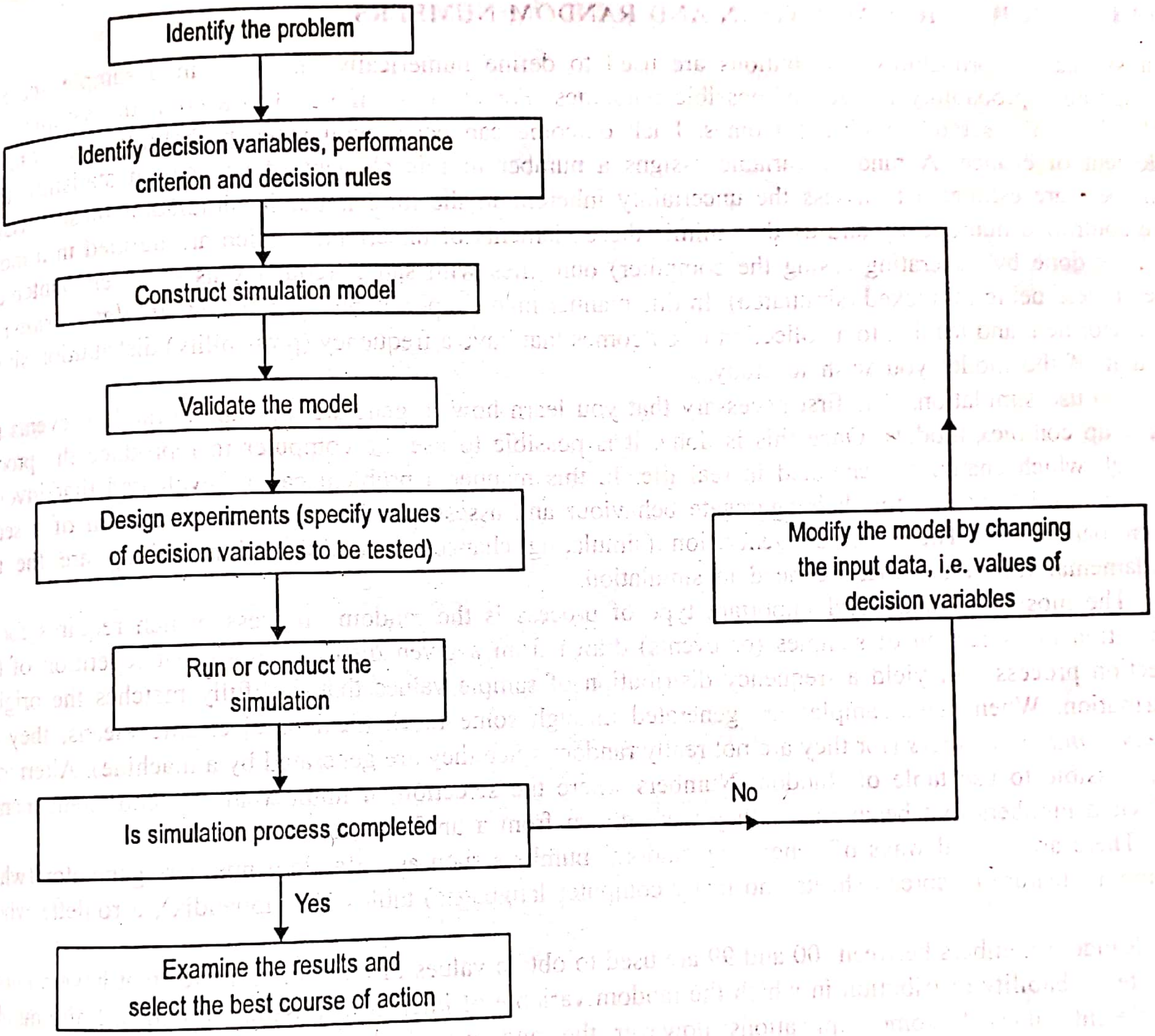


Fig. 19.1 Steps of Simulation Process

4. Simulation can be used as a pre-service test to try out new policies and decision rules for operating a system before running the risk of experimentation in the real system.
5. The only 'remaining tool' when all other techniques become intractable or fail.

Disadvantages

1. Sometimes simulation models are expensive and take a long time to develop. For example, a corporate planning model may take a long time to develop and prove expensive also.
2. It is the trial and error approach that produce different solutions in repeated runs. This means it does not generate optimal solutions to problems.
3. Each application of simulation is ad hoc to a great extent.
4. The simulation model does not produce answers by itself. The user has to provide all the constraints for the solutions which he wants to examine.

19.4 STOCHASTIC SIMULATION AND RANDOM NUMBERS

In simulation, probability distributions are used to define numerically outcomes in a sample space by assigning a probability to each of possible outcomes. For example, if you flip a coin, the sample space $\{H, T\}$ is the set of possible outcomes. Each outcome can occur with some probability, reflecting the element of chance. A random variable assigns a number to this element of chance. In statistics, these numbers are estimated to assess the uncertainty inherent in the model, but in simulation these variables are controlled numerically and used to mimic these elements of uncertainty which are defined in a model. This is done by generating (using the computer) outcomes with same frequency as those encountered in the process being mimicked (simulated). In this manner many experiments (also called *simulation runs*) can be performed, and leading to a collection of outcomes that have a frequency (probability) distribution similar to that of the model you wish to study.

To use simulation, it is first necessary that you learn how to generate the sample random events that make up complex models. Once this is done, it is possible to use the computer to reproduce the process through which chance is generated in real life. In this manner a problem can be evaluated that involves many interrelationships for their aggregate behaviour and assess this behaviour as a function of a set of given parameters. Thus process generation (simulating chance processes) and modelling are the two fundamental techniques that we need in simulation.

The most elementary and important type of process is the random process, which requires for its simulation the selection of samples (or events) drawn from a given distribution so that repetition of this selection process will yield a frequency distribution of sample values that faithfully matches the original distribution. When these samples are generated through some mechanical or electronic means, they are *pseudo random numbers* (for they are not really random since they are generated by a machine). Alternately it is possible to use table of Random Numbers where the selection of number in any consistent manner will yield numbers that behave as if they were drawn from a uniform distribution.

There are several ways of generating random numbers such as: Random numbers generator (which are inbuilt feature of spread sheets and many computer languages) tables (see appendix), a roulette wheel, etc.

Random numbers between 00 and 99 are used to obtain values of random variables that have a known discrete probability distribution in which the random variable of interest can assume one of a finite number of different values. In some applications, however, the random variables are continuous, that is, they can assume any real value according to a continuous probability distribution. For example, in queuing theory applications, the amount of time a server spends with a customer is such a random variable which might follow an exponential distribution.

19.4.1 Monte Carlo Simulation

The principle behind the Monte Carlo simulation technique is representative of the given system under analysis by a system described by some known probability distribution and then drawing random samples from probability distribution by means of random numbers. In case it is not possible to describe a system in terms of standard probability distribution such as normal, Poisson, exponential, gamma, etc. an empirical probability distribution can be constructed.

The Monte Carlo simulation technique consists of following steps:

(i) Setting up a probability distribution for variables to be analysed.
(ii) Building a cumulative probability distribution for each random variable.
(iii) Generate random numbers. Assign an appropriate set of random numbers to represent value or range (interval) of values for each random variable.

(iv) Conduct the simulation experiment by means of random sampling.
(v) Repeat Step 4 until the required number of simulation runs has been generated.
(vi) Design and implement a course of action and maintain control.

19.4.2 Random Number Generation

Monte Carlo simulation requires the generation of a sequence of random numbers. This sequence of random numbers help in choosing random observations (samples) from the probability distribution.

(a) *Arithmetic computation* The *n*th random number r_n consisting of *k*-digits generated by using multiplicative congruential method is given by

$$r_n \equiv p.r_{n-1} \text{ (modulo } m)$$

Where p and m are positive integers, $p < m$, r_{n-1} is a *k*-digit number and modulo m means that r_n is the remainder when $p.r_{n-1}$ is divided by m. This means, r_n and $p.r_{n-1}$ differ by an integer multiple of m. To start the process of generating random numbers, the first random number (also called *seed*) r_0 is specified by the user. Then using above recurrence relation a sequence of *k*-digit random number with period $h < m$ at which point the number r_0 occurs again can be generated.

For illustration, let $p = 35$, $m = 100$ and arbitrarily start with $r_0 = 57$. Since $m - 1 = 99$ is the 2-digit number, therefore, it will generate 2-digit random numbers:

$$r_1 = p\,r_0 \text{ (modulo } m) = 35 \times 57 \text{ (modulo 100)}$$
$$= 1{,}995/100 = 95, \text{ remainder}$$
$$r_2 = p\,r_1 \text{ (modulo } m) = 35 \times 95 \text{ (modulo 100)}$$
$$= 3{,}325/100 = 25, \text{ remainder}$$
$$r_3 = p\,r_2 \text{ (modulo } m) = 35 \times 25 \text{ (modulo 100)}$$
$$= 875/100 = 75, \text{ remainder}$$

The choice of r_0 and p for any given value of m require great care, and the method used is also not a random process because sequence of numbers generated is determined by the input data for the method. Thus, the numbers generated through this process are *pseudo random numbers* because these are reproducible and hence, not random.

The above defined recurrence relation can also be used to generate random numbers as decimal fraction between 0 and 1 with a desired number of digits. For this, the recurrence relation $u_n = r_n/m$ is used to generate uniformly distributed decimal fraction between 0 and 1.

(b) ***Computer generator*** The random numbers that are generated by using computer software are uniformly distributed decimal fractions between 0 and 1. The software works on the concept of cumulative *distribution function* for the random variables for which we are seeking to generate random numbers.

For example, for the negative exponential function with density function $f(x) = \lambda e^{-\lambda x}$, $0 < x < \infty$, the cumulative distribution function is given by

$$F(x) = \int_0^x \lambda\, e^{-\lambda x}\, dx = 1 - e^{-\lambda x}$$

or
$$e^{-\lambda x} = 1 - F(x)$$

Taking logarithm on both sides, we have

$$-\lambda x = \log\,[1 - F(x)]$$

or
$$x = -\,(1/\lambda)\log\,[1 - F(x)]$$

If $r = F(x)$ is a uniformly distributed random decimal fraction between 0 and 1, then the exponential variable associated with r is given by

$$x_n = -(1/\lambda)\log(1-r) = -(1/\lambda)\log r.$$

This is an exponential process generator since $1 - r$ is a random number and can be replaced by r.

Remark While picking up random numbers from the random number table, the starting point on the table is immaterial. That is, we may start with any number in any column or row, and proceed in the same column or row to the next number, but a consistent, unvaried (i.e. we should not jump from one number to another indiscriminately) pattern should be followed in drawing random numbers. If random numbers are to be taken for more than one concerned variables, then different random numbers for each variable should be used because same random numbers could imply dependence among different variables.

A number of process generators for use with a digital computer are shown in Table 19.2.

Table 19.2 Some Process Generators

Theoretical probability distribution	*Parameters*	*Process generators for random variable, x*
(a) *Discrete Random Variables*		
Uniform	$a, b = x$	where $\frac{x-a}{b-a} < r \le \frac{x-a+1}{b-a+1}$; $a \le x \le b$, r = random number
Binomial	$n, p = \sum_{i=1}^{n} x_i$,	where $x_i = \begin{cases} 1, & r_i \le p \\ 0, & r_i > p \end{cases}$; p = prob. of success; n = number of trials
Poisson	$\lambda = k - 1$,	where $\sum_{i=1}^{k-1} \frac{-\log r_i}{\lambda} \le 1 \le \sum_{i=1}^{k} \frac{-\log r_i}{\lambda}$; λ = mean arrival rate per unit of time
(b) *Continuous Random Variables*		
Uniform	$a, b = a + (b, a)\,r$	
Exponential	$\lambda = (-1/\lambda)\log r$	
Normal	$\mu, \sigma, a, b = \begin{cases} a, & u \le a \\ u, & ha < u < b;\ u = [(-2\log r_1)^{1/2}(\cos 6.283\, r_2)\,\sigma + \mu)] \\ b, & u \ge b \end{cases}$	; μ = mean, σ = standard deviation

19.5 SIMULATION OF INVENTORY PROBLEMS

Example 19.1 Using random numbers to simulate a sample, find the probability that a packet of 6 products does not contain any defective product, when the production line produces 10 per cent defective products. Compare your answer with the expected probability.

[*ICWA, Dec. 1990*]

Solution Given that 10 per cent of the total production is defective and 90 per cent is non-defective. If we have 100 random numbers (0 to 99), then 90 or 90 per cent of them represent non-defective products and remaining 10 (or 10 per cent) of them represent defective products. Thus, the random numbers 00 to 89 are assigned to variables representing non-defective products and 90 to 100 are assigned to variables representing defective products.

If we choose a set of 2-digit random numbers in the range 00 to 99 to represent a packet of 6 products as shown below, then we would expect that 90 per cent of the time they would fall in the range 00 to 89.

Sample number	*Random number*					
A	86	02	22	57	51	68
B	39	77	32	77	09	79
C	28	06	24	25	93	22
D	97	66	63	99	61	80
E	69	30	16	09	05	53
F	33	63	99	19	87	26
G	87	14	77	43	96	43
H	99	53	93	61	28	52
I	93	86	52	77	65	15
J	18	46	23	34	25	85

Here it may be noted that out of ten simulated samples 6 contain one or more defectives and 4 contain no defectives. Thus, the expected percentage of non-defective products is 40 per cent. However, theoretically the probability that a packet of 6 products containing no defective product is $(0.9)^6 = 0.53144$ = 53.14%.

Example 19.2 A bakery keeps stock of a popular brand of cake. Previous experience shows the daily demand pattern for the item with associated probabilities, as given below:

Daily demand (number) :	0	10	20	30	40	50
Probability :	0.01	0.20	0.15	0.50	0.12	0.02

Use the following sequence of random numbers to simulate the demand for next 10 days.

Random numbers: 25, 39, 65, 76, 12, 05, 73, 89, 19, 49.

Also estimate the daily average demand for the cakes on the basis of simulated data.

[ICWA, Dec. 1986]

Solution Using the daily demand distribution, we obtain a probability distribution as shown in Table 19.2.

Table 19.2 Daily Demand Distribution

Daily demand	*Probability*	*Cumulative probability*	*Random number interval*
0	0.01	0.01	00
10	0.20	0.21	01 – 20
20	0.15	0.36	21 – 35
30	0.50	0.86	36 – 85
40	0.12	0.98	86 – 97
50	0.02	1.00	98 – 99

Conduct the simulation experiment for demand by taking a sample of 10 random numbers from a table of random numbers, which represent the sequence of 10 samples. Each random sample number here is a sample of demand.

The simulation calculations for a period of 10 days are given in Table 19.3.

Table 19.3 Simulation Experiments

Days	*Random number*	*Demand*	
1	40	30	because 0.36 < 0.40 < 0.85
2	19	10	because 0.01 < 0.19 < 0.20,
3	87	40	and so on
4	83	30	
5	73	30	
6	84	30	
7	29	20	
8	09	10	
9	02	10	
10	20	10	
	Total	= 220	

Expected demand = 220/10 = 22 units per day

Example 19.3: A book store wishes to carry a particular book in stock. Demand is probabilistic and replenishment of stock takes 2 days (i.e. if an order is placed on March 1, it will be delivered at the end of the day on March 3). The probabilities of demand are given below:

Demand (daily) :	0	1	2	3	4
Probability :	0.05	0.10	0.30	0.45	0.10

Each time an order is placed, the store incurs an ordering cost of Rs 10 per order. The store also incurs a carrying cost of Re 0.05 per book per day. The inventory carrying cost is calculated on the basis of stock at the end of each day. The manager of the book store wishes to compare two options for his inventory decision.

A : Order 5 books when the inventory at the beginning of the day plus orders outstanding is less than 8 books.

B : Order 8 books when the inventory at the beginning of the day plus orders outstanding is less than 8.

Currently (beginning of 1st day) the store has a stock of 8 books plus 6 books ordered two days ago and expected to arrive next day. Using Monte Carlo simulation for 10 cycles, recommend which option the manager should choose.

The two digit random numbers are:

89, 34, 78, 63, 61, 81, 39, 16, 13, 73 [*ICWA, June 1988*]

Solution Using the daily demand distribution, we obtain a probability distribution as shown in Table 19.4.

Table 19.4 Daily Demand Distribution

Daily demand	*Probability*	*Cumulative probability*	*Random number interval*
0	0.05	0.05	00 – 04
1	0.10	0.15	05 – 14
2	0.30	0.45	15 – 44
3	0.45	0.90	45 – 89
4	0.10	1.00	90 – 99

Given that stock in hand is of 8 books and stock on order is 5 books (expected next day)

Table 19.5 Option A

Random number	*Demand daily*	*Closing stock in hand*	*Receipt*	*Opening stock in hand*	*Stock on order*	*Order quantity*	*Closing stock*
89	3	8	–	8 – 3 = 5	6	–	6
34	2	5	6	6 + 5 – 2 = 9	–	–	–
78	3	9	–	9 – 3 = 6	–	5	5
63	3	6	–	6 – 3 = 3	5	–	5
61	3	3	–	3 – 3 = 0	5	5	10
81	3	0	5	5 – 3 = 2	5	5	10
39	2	2	–	2 – 2 = 0	10	–	10
16	2	0	5	5 – 2 = 3	5	–	5
13	1	3	5	5 + 3 – 1 = 7	0	5	5
73	3	7	–	7 – 3 = 4	5	–	5

Since 5 books have been ordered four times as shown in Table 19.5, therefore, total ordering cost is Rs (4 × 10) = Rs 40.

Closing stock of 10 days is of 39 (= 5 + 9 + 6 + 3 + 2 + 3 + 7 + 4) books. Therefore, the holding cost at the rate of Re 0.5 per book per day is Rs (39 × 0.5) = Rs 19.5.

Total cost for 10 days = Ordering cost + Holding cost = Rs (40 + 19.5) = Rs 59.5.

Table 19.6 Option B

Random number	*Demand daily*	*Closing stock in hand*	*Receipt*	*Opening stock in hand*	*Stock on order*	*Order quantity*	*Closing stock*
89	3	8	–	8 – 3 = 5	6	–	6
34	2	5	6	6 + 5 – 2 = 9	–	–	–
78	3	9	–	9 – 3 = 6	–	8	8
63	3	6	–	6 – 3 = 3	8	–	8
61	3	3	–	3 – 3 = 0	8	–	8
81	3	0	8	8 + 0 – 3 = 5	–	8	8
39	2	5	–	5 – 2 = 3	8	–	8
16	2	3	–	3 – 2 = 1	8	–	8
13	1	1	8	8 + 1 – 1 = 8	–	–	–
73	3	8	—	8 – 3 = 5	—	8	8

Since 8 books have been ordered three times as shown in Table 19.6 when the inventory of books at the beginning of the day plus orders outstanding is less than 8. Therefore, total ordering cost is: Rs (3 × 10) = Rs 30.

Closing stock of 10 days is of 45 (= 5 + 9 + 6 + 3 + 5 + 3 + 1 + 8 + 5) books. Therefore holding cost, Re 0.5 per book per day is Rs (45 × 0.5) = Rs 22.50.

Total cost for 10 days = Ordering cost + Holding cost = Rs 52.50. Since option B has lower total cost than option A, therefore manager should choose option B.

Example 19.4 XYZ spare parts company wishes to determine the levels of stock it should carry for the items in its range. Demand is not certain and there is a lead time for stock replenishment. For one item X, the following information is obtained:

Demand (units/day)	:	3	4	5	6	7
Probability	:	0.10	0.20	0.30	0.30	0.10
Carrying cost (per unit/day)	:	Rs 2				
Ordering cost (per order)	:	Rs 50				
Lead time for replenishment	:	3 days				

Stock on hand at the beginning of the simulation exercise was 20 units.

Carry out a simulation run over a period of 10 days with the objective of evaluating the inventory rule: *Order 15 units when present inventory plus any outstanding order falls below 15 units.*

The sequence of random numbers to be used is: 0, 9, 1, 1, 5, 1, 8, 6, 3, 5, 7, 1, 2, 9 using the first number for day one.

Solution Let us begin simulation by assuming that

(i) orders are placed at the end of the day and received after 3 days at the end of the day.

(ii) back orders are accumulated in case of short supply and are supplied when stock is available.

The cumulative probability distribution and the random number range for daily demand is shown in Table 19.7.

Table 19.7 Daily Demand Distribution

Daily demand	*Probability*	*Cumulative probability*	*Random number range*
3	0.10	0.10	00
4	0.20	0.30	01 – 02
5	0.30	0.60	03 – 05
6	0.30	0.90	06 – 08
7	0.10	1.00	09

The results of the simulation experiment conducted are shown in Table 19.8.

Table 19.8 Simulation Experiments

Days	Opening stock	Random number	Resulting demand	Closing stock	Order placed	Order delivered	Average stock in the evening
1	10	0	3	17	–	–	19.5
2	17	9	7	10	15	–	13.5
3	10	1	4	6	–	–	8
4	6	1	4	2	–	–	4
5	2	5	5	0 (– 3)*	15	15	1
6	12	1	4	8	–	–	10
7	8	8	6	2	–	–	6
8	2	6	6	0 (– 4)*	15	15	1
9	11	3	5	6	–	–	8.5
10	6	5	5	1	–	–	3.5

* Negative figure indicates back orders.

Average ending stock = 78/10 = 7.8 units/day

Daily ordering cost = (Cost of placing one order) × (Number of orders placed per day)
= 50 × 3 = Rs 150

Daily carrying cost = (Cost of carrying one unit for one day) × (Average ending stock)
= 2 × 7.8 = Rs 15.60

Total daily inventory cost = Daily ordering cost + Daily carrying cost = 150 + 15.60 = Rs 165.60.

Example 19.5 The manager of a warehouse is interested in designing an inventory control system for one of the products in stock. The demand for the product comes from numerous retail outlets and orders arrive on a weekly basis. The warehouse receives its stock from the factory but the lead time is not constant. The manager wants to determine the best time to release orders to the factory so that stockouts are minimised yet inventory holding costs are at acceptable levels. Any order from retailers not supplied on a given day constitute lost demand. Based on a sampling study, the following data are available.

Demand per week (in thousand)	Probability	Lead time	Probability
0	0.20	2	0.30
1	0.40	3	0.40
2	0.30	4	0.30
3	0.10		

The manager of the warehouse has determined the following cost parameters: ordering cost (C_0) per order equals Rs 50, carrying cost (C_h) equals Rs 2 per thousand units per week, and shortage cost (C_s) equals Rs 10 per thousand units.

The objective of inventory analysis is to determine the optimal size of an order and the best time to place an order. The following ordering policy has been suggested.

Policy: Whenever the inventory level becomes less than or equal to 2,000 units (reorder level), an order equal to the difference between current inventory balance and the specified maximum replenishment level is equal to 4,000 units is placed.

Simulate the policy for a week's period assuming that the (i) beginning inventory is 3,000 units, (ii) no back orders are permitted, (iii) each order is placed at the beginning of the week as so on as inventory level is less than or equal to the reorder level, and (iv) the replenishment orders are received at the beginning of the week.

Solution Using weekly demand and lead time distributions, assign an appropriate set of random numbers to represent value (range) of variables as shown in Tables 19.9 and 19.10, respectively.

Table 19.9 Probabilities and Random Number Interval for Weekly Demand

Weekly demand (in thousand)	*Probability*	*Cumulative probability*	*Random number interval*
0	0.20	0.20	00 – 19
1	0.40	0.60	20 – 59
2	0.30	0.90	60 – 89
3	0.10	1.00	90 – 99

Table 19.10 Probabilities and Random Number Interval for Lead Time

Lead time (weeks)	*Probability*	*Cumulative probability*	*Random number interval*
2	0.30	0.30	00 – 29
3	0.40	0.70	30 – 69
4	0.30	1.00	70 – 99

The simulation experiment conducted for 10 weeks period is shown in Table 19.11. The simulation process begins with an inventory level of 3,000 units. The following four steps occur in the simulation process.

1. Begin each simulation week by checking whether any order has just arrived. If it has, increase the beginning (current) stock (inventory) by the quantity received.

2. Generate a weekly demand from the demand probability distribution in Table 19.9 by selection of a random number. This random number is recorded in column 4. The demand simulated is recorded in column 5.

The random number 31 generates a demand of 1,000 units when it is subtracted from the initial inventory level value of 3,000 units, yields an ending inventory of 2,000 units at the end of the first week.

3. Compute the ending inventory every week and record it in column 7.

$$\text{Ending inventory} = \text{Beginning inventory} - \text{Demand} = 3{,}000 - 1{,}000 = 2{,}000$$

If on hand inventory is not sufficient to meet the week's demand, then record the number of units short in column 6.

4. Determine whether the week's ending inventory has reached the reorder level. If it has, and if there is no outstanding (back orders), then place an order.

Since ending inventory of 2,000 units is equal to the reorder level, therefore, an order for 4,000 – 2,000 = 2,000 units is placed.

5. The lead time for the new order is simulated by first choosing a random number and recording it in column 8. Finally, this random number is converted into a lead time (column 9) by using the lead time distribution in Table 19.10.

The random number 29 corresponds to a lead time of 2 weeks. With 2,000 units to be held (carried) in stock, therefore the holding cost of Rs 4 is paid and since there were no shortages, there is no shortage cost. Summing these cost yields a total inventory cost (column 10) for week one of Rs 54.

The same step-by-step process is repeated for the remaining 10 weeks of the simulation experiment.

Analysis of Inventory Cost

Average ending inventory $= \dfrac{1{,}000 \text{ total unit}}{10 \text{ weeks}} = 100$ units per week.

Average number of orders placed $= \dfrac{2 \text{ orders}}{10 \text{ weeks}} = 0.2$ order per week.

Average number of lost sales $= \dfrac{7{,}000}{1{,}000} = 7$ units per week.

Total average inventory cost = Ordering cost + Holding cost + Shortage cost
= (Cost of placing one order) × (Number of orders placed per week) + (Cost of holding one unit for one week) × (Average ending inventory) + (Cost per lost sale) × (Average number of lost sales per week)

$$= \frac{100}{10} + \frac{16}{10} + \frac{70}{10} = 10 + 1.6 + 7 = \text{Rs. } 18.6$$

Table 19.11 Inventory Simulation Experiments

Maximum inventory level = 4,000 units | *Reorder level = 2,000 units*

Week	*Order receipt*	*Beginning inventory*	*Random number*	*Demand*	*Ending inventory*	*Quantity ordered*	*Random number*	*Lead time*	*Total cost (TC)* C_0	+ C_h	+ C_s	*= TC (Rs)*
1	0	3,000	31	1,000	2,000	2,000	29	2	50	4	–	= 54
2	0	2,000	70	2,000	0	0	–	–	–	–	–	–
3	0	0	53	1,000	(– 1,000)	0	–	–	0	0	10	= 10
4	2,000	2,000	86	2,000	0	4,000	83	4	50	–	–	= 50
5	0	0	32	1,000	(– 1,000)	0					10	= 10
6	0	0	78	2,000	(– 2,000)	0					20	= 20
7	0	0	26	1,000	(– 1,000)	0					10	= 10
8	0	0	64	2,000	(– 2,000)	0					20	= 20
9	4,000	4,000	45	1,000	3,000	0				6	–	= 06
10	0	3,000	12	0	3,000	0				6	–	= 06
			Total		1,000				100	16	70	

The negative figures in Table 19.11 enclosed in the bracket indicate loss of sales.

19.6 SIMULATION OF QUEUING PROBLEMS

Example 19.6 A dentist schedules all his patients for 30 minute appointments. Some of the patients take more or less than 30 minutes depending on the type of dental work to be done. The following summary shows the various categories of work, their probabilities and time actually needed to complete the work:

Category of service	*Time required (minutes)*	*Probability of category*
Filling	45	0.40
Crown	60	0.15
Cleaning	15	0.15
Extraction	45	0.10
Checkup	15	0.20

Simulate the dentist's clinic for four hours and determine the average waiting time for the patients as well as the idleness of the doctor. Assume that all the patients show up at the clinic at exactly their scheduled arrival time starting at 8.00 a.m. Use the following random numbers for handling the above problem: 40 82 11 34 25 66 17 79 [*CA, Nov. 1990*]

Solution The cumulative probability distribution and random number interval for service time are shown in Table 19.12.

Table 19.12

Category of service	*Service time required (minutes)*	*Probability*	*Cumulative probability*	*Random number interval*
Filling	45	0.40	0.40	00 – 39
Crown	60	0.15	0.55	40 – 54
Cleaning	15	0.15	0.70	55 – 69
Extraction	45	0.10	0.80	70 – 79
Checkup	15	0.20	1.00	80 – 99

The various parameters of a queuing system such as arrival pattern of customers, service time, waiting time in the context of the given problem are shown in Tables 19.13 to 19.15.

Table 19.13 Arrival Pattern and Nature of Service

Patient number	*Scheduled arrival*	*Random number*	*Category of service*	*Service time (minutes)*
1	8.00	40	Crown	60
2	8.30	82	Checkup	15
3	9.00	11	Filling	45
4	9.30	34	Filling	45
5	10.00	25	Filling	45
6	10.30	66	Cleaning	15
7	11.00	17	Filling	45
8	11.30	79	Extraction	45

Table 19.14 Computation of Arrivals, Departures and Waiting of Patients

Time	*Event (Patient number)*	*Patient number (Time to exit)*	*Waiting (Patient number)*
8.00	1 arrive	1 (60)	–
8.30	2 arrive	1 (30)	2
9.00	1 departs; 3 arrive	2 (15)	3
9.15	2 depart	3 (45)	–
9.30	4 arrive	3 (30)	4
10.00	3 depart; 5 arrive	4 (45)	5
10.30	6 arrive	4 (15)	5, 6
10.45	4 depart	5 (45)	6
11.00	7 arrive	5 (30)	6, 7
11.30	5 depart; 8 arrive	6 (15)	7, 8
11.45	6 depart	7 (45)	8
12.00	End	7 (30)	8

The dentist was not idle during the entire simulated period. The waiting times for the patients were as follows:

Table 19.15 Computation of Average Waiting Time

Patient	*Arrival time*	*Service starts at*	*Waiting time (minutes)*
1	8.00	8.00	0
2	8.30	9.00	30
3	9.00	9.15	15
4	9.30	10.00	30
5	10.00	10.45	45
6	10.30	11.30	60
7	11.00	11.45	45
8	11.30	12.30	60
			280

The average waiting time = 280/8 = 35 minutes.

Example 19.7 A firm has a single channel service station with the following arrival and service time probability distributions:

Inter-arrival time (minutes)	*Probability*	*Service time (minutes)*	*Probability*
		5	0.08
10	0.10	10	0.14
15	0.25	15	0.18
20	0.30	20	0.24
25	0.25	25	0.22
30	0.10	30	0.14

Solution The cumulative probability distributions and random number interval for inter-arrival time and service time are shown in Table 19.19.

Table 19.19

Arrival time		*Cumulative probability*	*Random number interval*	*Service time*		*Cumulative probability*	*Random number interval*
Minutes	*Probability*			*Minutes*	*Probability*		
2	0.15	0.15	00 – 14	1	0.10	0.10	00 – 09
4	0.23	0.38	15 – 37	3	0.22	0.32	10 – 31
6	0.35	0.73	38 – 72	5	0.35	0.67	32 – 66
8	0.17	0.90	73 – 89	7	0.23	0.90	67 – 89
10	0.10	1.00	90 – 99	9	0.10	1.00	90 – 99

The simulation work sheet developed for the given problem is shown in Table 19.20.

Table 19.20

Random number (1)	*Inter-arrival time (min.)*	*Arrival time (min.)*	*Service starts (min.)*	*Random number (2)*	*Service time (min.)*	*Service ends (min.)*	*Waiting time Attendant (min.)*	*Waiting time Customer (min.)*	*Line length*
93	10	9.10	9.10	71	7	9.17	10	–	–
14	2	9.12	9.17	63	5	9.22	–	5	1
72	6	9.18	9.22	14	3	9.25	–	4	1
10	2	9.20	9.25	53	5	9.30	–	5	1
21	4	9.24	9.30	64	5	9.35	–	6	1
81	8	9.32	0.35	42	5	9.40	–	3	1
87	8	9.40	9.40	07	1	9.41	–	–	–
90	10	9.50	9.50	54	5	9.55	9	–	–
38	6	9.56	9.56	66	5	10.01	1	–	–
Total	56				41		20	23	5

(i) Average queue length = 5/9 = 0.56 = 1 customer (approx.)
(ii) Average waiting time of customer before service = 23/9 = 2.56 minutes.
(iii) Average service idle time = 20/9 = 2.22 minutes.
(iv) Average service time = 41/9 = 4.56 minutes
(v) Time a customer spends in the system = (4.56 + 2.56) = 7.12 minutes.
(vi) Percentage of service idle time = 20/(20 + 41) = 0.33.

19.7 SIMULATION OF INVESTMENT PROBLEMS

Example 19.9 The Investment Corporation wants to study the investment projects based on three factors: market demand in units; price per unit minus cost per unit and investment required. These factors are felt to be independent of each other. In analysing a new consumer product, the Corporation estimates the following probability distributions:

Annual demand		*Price minus cost per unit*		*Investment required*	
Units	*Probability*	*Rs*	*Probability*	*Rs*	*Probability*
20,000	0.05	3.00	0.10	17,50,000	0.25
25,000	0.10	5.00	0.20	20,00,000	0.50
30,000	0.20	7.00	0.40	25,00,000	0.25
35,000	0.30	9.00	0.20		
40,000	0.20	10.00	0.10		
45,000	0.10				
50,000	0.05				

Using simulation process, repeat the trial 10 times, compute the return on investment for each trial taking these three factors into account. What is the most likely return?

Solution The return per annum can be computed by the following expression

$$\text{Return (R)} = \frac{(\text{Price} - \text{Cost}) \times \text{Number of units demanded}}{\text{Investment}}$$

Developing a cumulative probability distribution corresponding to each of the three factors, an appropriate set of random numbers is assigned to represent each of the three factors as shown in Tables 19.21, 19.22 and 19.23.

Table 19.21

Annual demand	*Probability*	*Cumulative probability*	*Random number*
20,000	0.05	0.05	00 – 14
25,000	0.10	0.15	05 – 14
30,000	0.20	0.35	15 – 34
35,000	0.30	0.65	35 – 64
40,000	0.20	0.85	65 – 84
45,000	0.10	0.95	85 – 94
50,000	0.05	1.00	95 – 99

Table 19.22

Price minus cost per unit	*Probability*	*Cumulative probability*	*Random number*
3.00	0.10	0.10	00 – 09
5.00	0.20	0.30	10 – 19
7.00	0.40	0.70	20 – 69
9.00	0.20	0.90	70 – 89
10.00	0.10	1.00	90 – 99

Table 19.23

Investment required	*Probability*	*Cumulative probability*	*Random number*
17,50,000	0.25	0.25	00 – 24
20,00,000	0.50	0.75	25 – 74
25,00,000	0.25	1.00	75 – 99

The simulation worksheet is prepared for 10 trials. The simulated return (R) is also calculated by using the formula for R as stated before. The results of simulation are shown in Table 19.24.

Table 19.24

Trials	*Random number for demand*	*Simulated demand ('000)*	*Random number for profit (price – cost) per unit*	*Simulated profit*	*Random number for investment*	*Simulated investment ('000)*	*Simulated return (%): Demand* $\times \frac{\text{Profit per unit}}{\text{Investment}} \times 100$
1	28	30	19	5.00	18	1750	8.57
2	57	35	07	3.00	61	2000	5.25
3	60	35	90	10.00	16	1750	20.00
4	17	30	02	3.00	71	2000	4.50
5	64	35	57	7.00	43	2000	12.25
6	20	30	28	5.00	68	2000	7.50
7	27	30	29	5.00	47	2000	7.50
8	58	35	83	9.00	24	1750	19.00
9	61	35	58	7.00	19	1750	14.00
10	30	30	41	7.00	97	2500	8.40

As shown in Table 19.24, the highest likely return is 20 per cent which corresponds to annual demand of 35,000 units yielding a profit of Rs 10 per unit and investment required is Rs 17,50,000.

19.8 SIMULATION OF MAINTENANCE PROBLEMS

Example 19.10 A plant has a large number of similar machines. The machine breakdowns or failures are random and independent.

The shift in-charge of the plant collected the data about the various machines breakdown times and the repair time required on hourly basis, and the record for the past 100 observations as shown below was:

Time between recorded machine breakdowns (hours)	*Probability*	*Repair time required (hours)*	*Probability*
0.5	0.05	1	0.28
1	0.06	2	0.52
1.5	0.16	3	0.20
2	0.33		
2.5	0.21		
3	0.19		

For each hour that one machine is down due to being or waiting to be repaired, the plant loses Rs 70 by way of lost production. A repairman is paid at Rs 20 per hour.

(a) Simulate this maintenance system for 15 breakdowns.
(b) How many repairmen should the plant hire for repair work.

Solution The random numbers coding for the hourly breakdowns and the repair times are shown in Tables 19.25 and 19.26.

Table 19.25 Random Number Coding for Breakdowns

Time between breakdowns(hours)	*Probability*	*Cumulative probability*	*Random number range*
0.5	0.05	0.05	00 – 04
1	0.06	0.11	05 – 10
1.5	0.16	0.27	11 – 26
2	0.33	0.60	27 – 59
2.5	0.21	0.81	60 – 80
3	0.19	1.00	81 – 99

Table 19.26 Random Number Coding for Repairs

Repair time required (hours)	*Probability*	*Cumulative probability*	*Random number range*
1	0.28	0.28	00 – 27
2	0.52	0.80	28 – 79
3	0.20	1.00	80 – 99

The simulation worksheet is shown in Table 19.27. It is assumed that the first day begins at midnight (00.00 hours) and also the repairman begins work at 00.00 hours. The first breakdown occurred at 2.30 A.M. and the second occurred after 3 hours at clock time of 5.30 A.M..

Total current maintenance cost = Idle time cost + Repairman's wage
+ (Repair time + Waiting time) × Hourly rate + Total hours × Hourly wages
= 57.30 × 70 + 38.30 × 20 = Rs 4,777

Maintenance Cost with Additional Repairman

If the plant hires two more repairmen, then no machine will wait for its repair. Thus, total idle time would be only the repairing time of 36.00 hours. Therefore,

Total cost = 36 × 70 + (38.30 × 2) × 20 = Rs 4,052

This shows that hiring more than two repairmen would only increase the total maintenance cost. Hence, the plant may hire one additional repairman.

Table 19.27 Simulation Worksheet

Breakdown number	*Random number for break-downs*	*Time between break-downs*	*Time of break-down*	*Repair work begins at*	*Random number for repair time*	*Repair time required*	*Repair work ends at*	*Total idle time (hours)*	*Waiting time (hours)*
(1)	*(2)*	*(3)*	*(4)*	*(5)*	*(6)*	*(7)*	*(8)*	*(9)*	*(10)*
1	61	2.5	02.30	02.30	87	3	05.30	3.00	–
2	85	3	05.30	05.30	39	2	07.30	2.00	–
3	16	1.5	07.00	07.30	28	2	09.30	2.30	0.30
4	46	2	09.00	09.30	97	3	12.30	3.30	0.30
5	88	3	12.00	12.30	69	2	14.30	2.30	0.30
6	08	1	13.00	14.30	87	3	17.30	4.30	1.30
7	82	3	16.00	17.30	52	2	19.30	3.30	1.30
8	56	2	18.00	19.30	52	2	21.30	3.30	1.30
9	22	1.5	19.30	21.30	15	1	22.30	3.00	2.00
10	49	2	21.30	22.30	85	3	01.30	4.00	1.00
11	44	2	23.30	01.30	41	2	03.30	4.00	2.00
12	33	2	01.30	03.30	82	3	06.30	5.00	2.00
13	77	2.5	04.00	06.30	98	3	09.30	5.30	2.30
14	87	3	07.00	09.30	99	3	12.30	5.30	2.30
15	54	2	09.00	12.30	23	2	14.30	5.30	3.30
				38.30		36.00		57.30	21.30

19.9 SIMULATION OF PERT PROBLEMS

Example 19.11 A project consists of eight activities A to H. The completion time for each activity is a random variable. The data concerning probability distribution along with completion times for each activity is as follows:

Activity	*Immediate predecessor(s)*	*Time (day)/Probability*								
		1	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*
A	–	–	–	–	0.2	–	0.4	0.4	–	–
B	–	–	–	–	–	–	0.5	–	0.5	–
C	A	–	–	0.7	0.3	–	–	–	–	–
D	B, C	–	–	–	–	0.9	–	–	0.1	–
E	A	–	–	–	–	0.2	–	–	–	0.8
F	D, E	–	–	–	0.6	0.4	–	–	–	–
G	E	–	–	0.4	0.4	–	0.2	–	–	–
H	F	–	0.4	–	–	–	–	0.6	–	–

(a) Draw the network diagram and identify the critical path using the expected activity times.

(b) Simulate the project to determine the activity times. Determine the critical path and project expected completion time.

(c) Repeat the simulation four times and state estimated duration of the project in each of the trials.

Solution (a) The network diagram based on the precedence relationships is shown in Fig. 19.2. The expected completion time of each activity is obtained by using the formula:

$$\text{Expected time} = \Sigma\,(\text{Activity time} \times \text{Probability})$$
$$= 4 \times 0.2 + 6 \times 0.4 + 7 \times 0.4 = 6 \text{ days (activity A)}$$

The critical path of the project is: 1 – 2 – 3 – 4 – 5 – 6 – 7, with expected completion time of 23.6 days. The random number coding for each of the activities expected time is shown in Table 19.28.

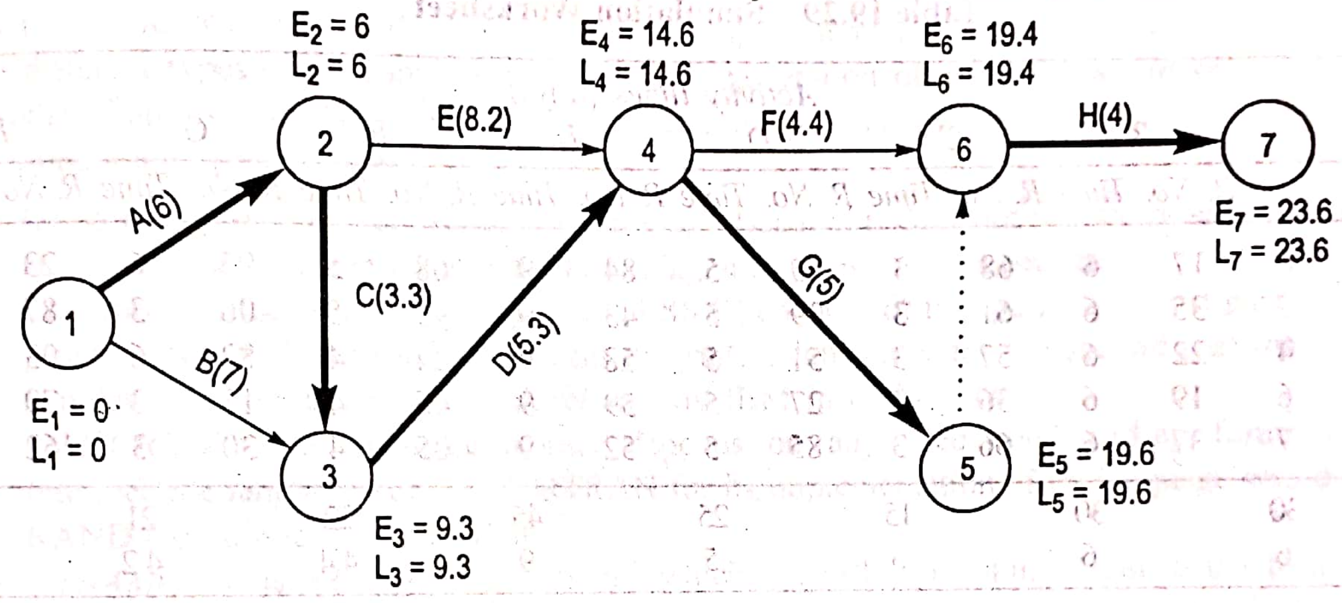


Fig. 19.2 Network Diagram

Table 19.28 Random Number Coding for Activity Times

Activity	*Time*	*Probability*	*Cumulative probability*	*Random number range*
A	4	0.20	0.20	00 – 19
	6	0.40	0.60	20 – 59
	7	0.40	1.00	60 – 99
B	6	0.50	0.50	00 – 49
	8	0.50	1.00	50 – 99
C	3	0.70	0.70	00 – 69
	4	0.30	1.00	70 – 99
D	5	0.90	0.90	00 – 89
	8	0.10	1.00	90 – 99
E	5	0.20	0.20	00 – 19
	9	0.80	1.00	20 – 99
F	4	0.60	0.60	00 – 59
	5	0.40	1.00	60 – 99
G	3	0.40	0.40	00 – 39
	4	0.40	0.80	40 – 79
	6	0.20	1.00	80 – 99
H	2	0.40	0.40	00 – 39
	7	0.60	1.00	40 – 99

The simulation worksheet for four simulation runs is shown in Table 19.29. For each run the project time is obtained as follows:

Total time = Larger of times for activities A, B and C + Larger of times for activities D and E + Larger of times for activities F and G + Time for activity H.

Using the data given in Table 19.29, we have the simulation results shown in Table 19.30.

Table 19.29 Simulation Worksheet

Run	*A*		*B*		*C*		*D*		*E*		*F*		*G*		*H*	
	R. No.	*Time*	*R. No.*	*Time*	*R. No.*	*Time*	*R. No.*	*Time*	*R. No.*	*Time*	*R. No.*	*Time*	*R. No.*	*Time*	*R. No.*	*Time*
1	22	6	17	6	68	3	65	5	84	9	68	5	95	6	23	2
2	92	7	35	6	61	3	09	5	43	9	95	5	06	3	87	7
3	02	4	22	6	57	3	51	5	58	9	24	4	82	6	03	2
4	47	6	19	6	36	3	27	5	59	9	46	4	13	3	79	7
5	93	7	37	6	66	3	85	5	52	9	05	4	30	3	62	7
Total		30		30		15		25		45		22		21		25
Average		6		6		3		5		9		4.4		4.2		5

Table 19.30 Simulation Results

Simulation run	*Activity time*	*Project duration (days)*	*Longest (critical) path*
1	6 + 9 + 6 + 2	23	1 – 2 – 3 – 4 – 5 – 6 – 7
			1 – 3 – 4 – 5 – 6 – 7
2	7 + 9 + 5 + 7	28	1 – 3 – 4 – 6 – 7
3	6 + 9 + 6 + 2	23	1 – 2 – 3 – 4 – 5 – 6 – 7
4	6 + 9 + 4 + 7	26	1 – 2 – 3 – 4 – 6 – 7
			1 – 3 – 4 – 6 – 7
5	7 + 9 + 4 + 7	27	1 – 3 – 4 – 6 – 7
		127	

Here it may be noted that simulated mean project completion time, 25.4 days is almost two days longer than the 23.6 days completion time indicated using expected values alone.

19.10 ROLE OF COMPUTERS IN SIMULATION

The role of computers in simulation is vital. They are used to generate random numbers, simulate the given problem with varying values of variables in few minutes and help the decision-maker to prepare reports which enable him to make decisions quickly as well as draw valid conclusions.

Computer languages available to help the simulation process can be divided into two categories:

19.10.1 General Purpose Programming Languages

The general purpose programming languages include FORTRAN, BASIC, COBOL, PL/I, Pascal, etc. To use these languages for simulation process an extensive programming experience is required. As can be seen, even in a simple queuing problem, many tedious details are involved in a simulation model.

19.10.2 Special Purpose Simulation Languages

Special simulation languages have few advantages such as: (i) They reduce programme preparation time and cost with features specially designed for simulation model. Such features generally include a master sequencing routine to automatically maintain an event sequence and to keep track of simulated time sub-routines to handle arrivals and departures in a queuing system; (ii) They have the capability to readily generate different types of random variates automatic generation of certain types of statistical tables, and various other features. (iii) They require little or no prior programming knowledge for use. Major special purpose simulation languages are

(i) *GPSS (General Purpose System Simulation)* Usually, it does not require programme writing. The system model is constructed via block diagrams using block commands.
The third version of this language, i.e. GPSS III consists of two parts. The first part is an assembly programme which converts the system descriptors into input for the second part that performs the simulation. This language was developed by IBM in early 1960s.

(ii) *SIMSCRIPT* This language neither depends on any predefined coding forms nor on any intermediate language such as FORTRAN for its implementation. This language was developed by RAND Corporation in early 1960s.

(iii) *DYNAMO* It is a computer programme which is capable of taking input in the form of a set of equations describing the system. These equations are evaluated continuously for each time interval to understand the behaviour of the system. This language was developed at MIT in 1959 and is best suited for econometric modelling of industrial complexes, urban, social and world systems planning.

The choice of a simulation package depends mainly on the specific purpose, the availability of simulation languages on a particular computer, the training and experience in simulation modelling and programming, and the availability of experienced programmers.

19.11 APPLICATIONS OF SIMULATION

There is a wide range of applications of computer-based simulation models because it is an approach rather than an application of specific techniques.

The major use of computer-based Monte-Carlo simulation model has been in the solution of complex queuing problems.

A number of job shop simulation programmes have been developed involving deterministic times for the individual operations of a given order. Due to different processing times for similar operations and different order operations sequences, it is difficult to predict the waiting time for a particular job at any given work centre. For better scheduling, orders must be scheduled with a provision of waiting at the various work centres they will pass through. Simulation can help in estimating accurately such waiting times.

A good deal of work has been done in the development of inventory simulation models such as determination of optimal reorder level and lot size under conditions of probabilistic demand and lead time, optimal review period and ordering policy for continuous review inventory models.

A number of network simulation models have also been developed. For example, with a randomly selected activity times the critical path can be evaluated. Repeating this process many times, the probability distribution of project completion time can be obtained as well as the probability that each given activity is on the critical path.

Decision Theory

13.1 INTRODUCTION

So far, we have studied models for decision-making under conditions of certainty and uncertainty in the environment. In Chapters 2 to 8, we have considered models where different parameters were assumed to be known with certainty, while in others, uncertainty, wherever occurring, was incorporated into the analysis by specifying the probability distributions that the variables under consideration follow. We now turn to a specialised area, known as *decision theory*, which provides a formal analytic framework for decision-making under conditions of uncertainty.

The decision theory, also called the *decision analysis*, is used to determine optimal strategies where a decision-maker is faced with several decision alternatives and an uncertain, or risky, pattern of future events. To recapitulate, all decision-making situations are characterised by the fact that two or more alternate courses of action are available to the decision-maker to choose from. Further, a decision may be defined as the selection by the decision-maker of an act, considered to be best according to some predesignated standard, from among the available options. The decision-making process, thus, involves the following steps:

(a) Identification of the various possible outcomes, called *states of nature* or *events*, E_i's, for the decision problem. The events are beyond the control of the decision-maker.

(b) Identification of all the courses of action, A_j's, or the strategies that are available to the decision-maker. The decision-maker has control over choice of these.

(c) Determination of the pay-off function which describes the consequences resulting from the different combinations of the acts and events. The pay-offs may be designated as V_{ij}'s—the pay-off resulting from *i*th event and *j*th strategy.

(d) Choosing from among the various alternatives on the basis of some criterion, which may involve the information given in step (c) only or which may require and incorporate some additional information.

We shall detail the decision analysis in three parts in this chapter. The first part deals with single stage decision making problems, where decisions are taken by considering the (monetary) pays-offs resulting from various combinations of alternative courses of action and outcomes possible. The second part considers multi-stage decision problems wherein multiple decisions need to be taken one after another. The idea in such cases is to choose the optimal sequence (of decisions) from among the various alternatives. Finally, the third part uses utility, instead of monetary pay-offs, as the criterion for decision-making. We consider these in turn now.

13.2 ONE-STAGE DECISION-MAKING PROBLEMS

As indicated, decision-making in case of single stage decision problems calls for (i) identification of the courses of action available to the decision-maker in the face of various possible events, (ii) developing a pay-off matrix, and (iii) choosing a particular course of action in accordance with some principle.

To understand the decision process under uncertain conditions, let us consider the following example.

Example 13.1 A bookstore sells a particular book of tax laws for Rs 100. It purchases the book for Rs 80 per copy. Since some of the tax laws change every year, the copies unsold at the end of a year become outdated and can be disposed of for Rs 30 each. According to past experience, the annual demand for this book is between 18 and 23 copies.

Assuming that the order for this book can be placed only once during the year, the problem before the store's manager is to decide how many copies of the book should be purchased for the next year.

For this problem, since the annual demand varies between 18 and 23 copies, there are six possible events:

E_1 :	18 copies are demanded,	E_4 :	21 copies are demanded,
E_2 :	19 copies are demanded,	E_5 :	22 copies are demanded,
E_3 :	20 copies are demanded,	E_6 :	23 copies are demanded.

Also, there are six possible strategies, or courses of action. They are:

A_1 :	buy 18 copies,	A_4 :	buy 21 copies,
A_2 :	buy 19 copies,	A_5 :	buy 22 copies,
A_3 :	buy 20 copies,	A_6 :	buy 23 copies.

Thus, in this problem there are 6 possible alternatives to choose from, and an equal number of states of nature, or events.

Having listed the possible acts and events, the next step is to construct the pay-off table.

13.2.1 Developing Pay-off and Regret Tables

A pay-off table depicts the economics of the given problem. A pay-off is a conditional value—a conditional profit, loss, or may be, a conditional cost. It is *conditional* in the sense that associated with each course of action is a certain profit/loss, given that certain event has occurred. Thus, the profit or loss resulting by the adoption of a certain strategy is dependent upon, and is therefore associated with, the particular event that may occur. A pay-off table thus represents the matrix of the conditional values associated with all the possible combinations of the acts and the events.

To consider how the pay-off table can be constructed for our example, let D denote the demand in units for the book, and let Q denote the quantity decided to be purchased (the course of action). The profit, P, for a total demand equal to, or greater than, the quantity purchased shall be equal to the difference of the total revenue from the sale of all the copies that were purchased and the total cost of procuring them. Thus, when $D \geq Q$, we have $P = 100Q - 80Q$ or $P = 20Q$. On the other hand, when the quantity demanded is less than the quantity purchased (i.e. $D < Q$), the profit shall be equal to the total revenue obtained from selling D copies *plus* the revenue from the unsold copies, equal to $30(Q - D)$, *minus* the total cost of buying Q copies. Thus, $P = 100D + 30(Q - D) - 80Q$ or $P = 70D - 50Q$. Alternatively, since each copy sold yields a profit of Rs 20 while every unsold copy involves a loss of Rs 50, the profit function, when $D < Q$, can be stated as: $P = 20D - 50(Q - D)$ which on simplification reduces to $P = 70D - 50Q$, the same as given above. Thus, we have

$$P = 20Q, \text{ when } D \geq Q,$$

and

$$P = 70D - 50Q, \text{ when } D < Q$$

The pay-offs for all combination A_j's and E_i's are given in Table 13.1.

TABLE 13.1 Pay-off Table

Event E_i	Act, A_j					
	$A_1 : 18$	$A_2 : 19$	$A_3 : 20$	$A_4 : 21$	$A_5 : 22$	$A_6 : 23$
$E_1 : 18$	360	310	260	210	160	110
$E_2 : 19$	360	380	330	280	230	180
$E_3 : 20$	360	380	400	350	300	250
$E_4 : 21$	360	380	400	420	370	320
$E_5 : 22$	360	380	400	420	440	390
$E_6 : 23$	360	380	400	420	440	460

Opportunity Loss or Regret Table The resultant outcomes of the various combinations of the acts and events (the states of nature) can alternatively be expressed in terms of the *opportunity loss*. Also called regret, the opportunity loss is defined as the amount of pay-off foregone by not adopting the optimal course of action—that which would give the highest pay-off, for each possible event. Thus, in the context of our example, the opportunity loss represents the amount of profit foregone by not stocking as many copies of the book as would yield the highest profit for each level of demand. For instance, if the demand is 18 copies, then the optimal act is to buy 18 copies for a profit of Rs 360. Note that in the first row, the highest profit is Rs 360 corresponding to the act A_1. If any other strategy is adopted, the profit earned would be less and, the greater the departure from the optimal strategy the lesser the profit earned and, consequently, the greater the opportunity loss (or regret). With the strategy of buying 20 copies, A_3, the profit is Rs 260 which is Rs 100 less than the profit that could be earned at this level of demand. Similarly, if the demand turns out for 20 copies, the optimal course would be to order 20 copies—the profit being Rs 400. Any order size other than this causes lesser profit than this. When, for example, 18 units, are stocked, the profit is Rs 360 which is Rs 40 less than the highest profit.

The pay-off matrix can be transformed into opportunity loss matrix by subtracting from the highest profit value in each row, all the other values in that row. The opportunity loss matrix is given in Table 13.2.

TABLE 13.2 Opportunity Loss or Regret Table

Event E_i	Act, A_j					
	$A_1 : 18$	$A_2 : 19$	$A_3 : 20$	$A_4 : 21$	$A_5 : 22$	$A_6 : 23$
$E_1 : 18$	0	50	100	150	200	250
$E_2 : 19$	20	0	50	100	150	200
$E_3 : 20$	40	20	0	50	100	150
$E_4 : 21$	60	40	20	0	50	100
$E_5 : 22$	80	60	40	20	0	50
$E_6 : 23$	100	80	60	40	20	0

Now, before proceeding to a discussion of the method of solution to this problem, a few observations follow. Although the pay-off matrix (or the opportunity loss matrix) serves as an adequate representation of many a practical business decision situation, all types of decision situations cannot be represented by this approach. In a subsequent section of this chapter (see: The Decision Trees) those situations are discussed where a sequence of decisions is involved. Such situations cannot usually be handled by this model. Further, sometimes the events judged are under the control of a rational opponent. For example, the outcome of a certain strategy by a businessman depends upon the strategy adopted by his business competitor, who can influence the result by his own action. In such cases, although the decision can be represented by a pay-off matrix, the solution procedure is quite different. Such decision situations are discussed in the chapter titled Theory of Games.

While constructing a pay-off matrix, the alternative courses of action and the possible outcomes (events) must be clearly determined. In any given situation, the listing of the events and the actions must be distinct, mutually exclusive and collectively exhaustive. Alternatives a_1 and a_2, for instance, cannot occur together. If their joint occurrence was possible, then it would be defined as another alternative a_3.

Besides, the pay-off matrix in respect of the given example indicates the profit obtainable under different action-event combinations, the pay-offs in some cases are required to be expressed in terms of cost (see Example 13.2).

13.2.2 Decision Rules

After setting up the pay-off table (or the opportunity loss table) we proceed to take the decision. There are several rules, or criteria, on the basis of which decision may be taken. The selection of an appropriate criterion depends on factors like the nature of decision situation, attitude of the decision-maker, and so on. We shall first discuss the decision rules for taking decisions in conditions of uncertainty and then for the conditions of risk.

A. **Decisions Under Uncertainty** The decision situations where there is no way in which the decision-maker can assess the probabilities of the various states of nature are called decisions under uncertainty. In such situations, the decision-maker has no idea at all as to which of the possible states of nature would occur nor has he a reason to believe why a given state is more, or less, likely to occur as another. With probabilities of the various outcomes unknown, the actual decisions are based on specific criteria. The several principles which may be employed for taking decisions in such conditions are discussed below. Their discussion in the context of the bookstore problem is obviously based on the assumption that the bookstore has only this information that the demand of the book varies between 18 and 23 inclusive, and no more.

(a) Laplace Principle

The Laplace Principle is based on the simple philosophy that if we are uncertain about the various events then we may treat them as equally probable. Under this assumption, the expected (mean) value of pay-off for each strategy is determined and the strategy with highest mean value is adopted. Of course, if the pay-offs are in terms of costs, we choose the strategy with the lowest average cost.

For Example 13.1, expected pay-offs for different acts are as follows:

Act	*Mean (Expected) Pay-off*
A_1	(360 + 360 + 360 + 360 + 360 + 360)/6 = Rs 360.0
A_2	(310 + 380 + 380 + 380 + 380 + 380)/6 = Rs 368.3
	(260 + 330 + 400 + 400 + 400 + 400)/6 = Rs 365.0

A_4	$(210 + 280 + 350 + 420 + 420 + 420)/6 = \text{Rs } 350.0$
A_5	$(160 + 230 + 300 + 370 + 440 + 440)/6 = \text{Rs } 323.3$
A_6	$(110 + 180 + 250 + 320 + 390 + 460)/6 = \text{Rs } 285.0$

Since the expected pay-off for A_2 is the maximum, it would be adopted. Thus, the bookstore manager would buy 19 copies of the book if he chooses to adopt the Laplace rule for taking decision.

(b) Maximin or Minimax Principle

This principle is adopted by pessimistic decision-makers who are conservative in their approach. Using this approach, the minimum pay-offs resulting from adoption of various strategies are considered and among these values the maximum one is selected. It involves, therefore, choosing the best (the maximum) profit from the set of worst (the minimum) profits.

When dealing with the costs, the maximum cost associated with each alternative is considered and the alternative which minimises this maximum cost is chosen. In this context, therefore, the principle used is *minimax*—the best (the minimum cost) of the worst (the maximum cost).

For our example, the minimum profit associated with various strategies is as follows:

A_1 : Rs 360	A_3 : Rs 260	A_5 : Rs 160
A_2 : Rs 310	A_4 : Rs 210	A_6 : Rs 110

Since the maximum of these is Rs 360, the strategy A_1 is selected corresponding to the maximin principle of choice.

(c) Maximax or Minimin Principle

The maximax principle is optimists' principle of choice. It suggests that for each strategy, the maximum profit should be considered and the strategy with which the highest of these values is associated should be chosen. The optimist obviously desires a chance for the maximum pay-off in the decision matrix.

For Example 13.1, the maximum pay-off associated with the different strategies is as follows:

A_1 : Rs 360	A_3 : Rs 400	A_5 : Rs 440
A_2 : Rs 380	A_4 : Rs 420	A_6 : Rs 460

The highest profit being Rs 460, strategy A_6 of ordering 23 copies of the book is the decision according to the maximax principle.

In decision problems dealing with costs, the minimum cost for each alternative is considered and then the alternative which minimises the minimum cost is selected. The principle in this case is obviously termed *minimin*.

(d) Hurwicz Principle

The Hurwicz principle of decision-making stipulates that a decision-maker's view may fall somewhere between the extreme pessimism of the maximin principle and the extreme optimism of the maximax principle. This principle provides a mechanism by which different levels of optimism and pessimism may be shown. For this, an index of optimism, α, is defined on scale ranging from 0 to 1. An $\alpha = 0$ indicates extreme pessimism while $\alpha = 1$ represents extreme optimism.

For taking a decision using this principle, first the decision-maker's degree of optimism is indicated on the scale. Assuming that the decision-maker is able to reflect a degree of optimism by assigning a particular value of α, we multiply the maximum profit for each strategy A_j by α, and the minimum profit for it by

$1-\alpha$. The sum of these products, called the *Hurwicz Criterion*, is obtained for each strategy and we select the alternative which maximises this quantity. Obviously, when $\alpha = 0$, only the minimum of the profits for each strategy would be considered and the decision would in effect be according to the maximin criterion. Similarly, when $\alpha = 1$, the decision would be identical to that arrived through the maximax principle.

Suppose that for Example 13.1, the decision-maker's degree of optimism is reflected adequately by $\alpha = 0.6$. With this, we can obtain Hurwicz Criterion associated with various courses of action as given in Table 13.3.

TABLE 13.3 Hurwicz Criterion for Various Acts

Act	*Max*	*Min*	*Criterion Value* = α*(Max Value)* + $(1-\alpha)$*(Min Value)*
A_1	360	360	$0.6 \times 360 + 0.4 \times 360 = 360$
A_2	380	310	$0.6 \times 380 + 0.4 \times 310 = 352$
A_3	400	260	$0.6 \times 400 + 0.4 \times 260 = 344$
A_4	420	210	$0.6 \times 420 + 0.4 \times 210 = 336$
A_5	440	160	$0.6 \times 440 + 0.4 \times 160 = 328$
A_6	460	110	$0.6 \times 460 + 0.4 \times 110 = 320$

Since the value associated with the strategy A_1 is the maximum, the decision is to choose this strategy under the given principle.

In the case of costs, the principle works like this. The minimum of the costs for each course of action is multiplied by α (the indicator of the degree of optimism of the decision-maker), and the maximum of the costs for each alternative is multiplied by $1-\alpha$. Then the sum of the products for each action strategy is obtained. The alternative for which the sum is the least is selected.

(e) Savage Principle

The Savage principle is based on the concept of regret and calls for selecting the course of action that minimises the maximum regret. It is alternatively known as the *principle of minimax regret*.

As a first step, the regret matrix is derived from the pay-off matrix, as explained earlier. Then the maximum regret value corresponding of each of the strategies is determined and the strategy which minimises the maximum regret is chosen. The principle of choice is also conservative in approach and is very close to the minimax principle applied to the original matrix containing pay-off values. It may, however, be noted that the results obtained by the two methods need not be the same.

From the regret matrix given in Table 13.2, we get the following maximum regret values associated with the various courses of action.

A_1:	Rs 100	A_3:	Rs 100	A_5:	Rs 200
A_2:	Rs 80	A_4:	Rs 150	A_6:	Rs 250

The maximum regret value for the strategy A_2 being the least, it represents the optimal choice.

B. Decision Under Risk The decision situations wherein the decision-maker chooses to consider several possible outcomes and the probabilities of their occurrence can be stated are called *decisions under risk*. The probabili... determined objectively from the past records. For Example 13.1, suppose that the bookstore observes from the past sales data that the proportion of times the number of copies sold is 18, 19, 20, 21, 22, and 23 are, respectively, 0.05, 0.10, 0.30, 0.40, 0.10 and 0.05. Of course, as the case here is, the sum of the probabilities must be 1.0—in keeping with the earlier observation that the set of possible outcomes in a decision problem should be mutually exclusive and collectively exhaustive.

However, past records may not be available in many cases to arrive at objective probabilities. In case the decision-maker may, on the basis of his experience and judgement, be able to assign subjective probabilities to the various outcomes, the problem can yet be solved as a decision problem under risk.

Under conditions of risk, there are generally two criteria to choose from. These are discussed below.

(a) Maximum Likelihood Principle

Under this principle, the decision-maker first considers the event that is most likely to occur. He then decides for the course of action which has the maximum conditional pay-off, corresponding to this event (of course, when the pay-off matrix is in terms of costs, then the action with the least conditional pay-off would be chosen). For our example, it is known that the probability (equal to 0.40) is the highest for a demand level of 21 copies. Therefore, we would consider the pay-offs resulting from adopting different strategies for this demand level, and observe that it is highest at Rs 420 when 21 copies are ordered for. Thus, the decision according to this criterion is to buy 21 copies of the book.

This principle may seem reasonable in many situations, specially where the probability of a particular event may be predominantly larger than the probabilities of the other possible events. However, it has the demerit that it ignores the available information, that is to say, the other possible events and their consequences are neglected.

(b) Expectation Principle

More generally, the decision-making in situations of risk is on the basis of the expectation principle. With the event probabilities assigned, objectively or subjectively as the case may be, the expected pay-off for each strategy is calculated by multiplying the pay-off values with their respective probabilities and then adding up these products. The strategy with the highest expected pay-off represents the optimal choice. It goes without saying that in problems involving pay-off matrix in terms of costs, optimal strategy is that corresponding to which the expected value is the least.

Symbolically, for a decision problem involving n events and m strategies, the expected pay-offs, EP, can be expressed as under:

$$EP_j = \sum_{i=1}^{n} p_i a_{ij} \qquad j = 1, 2, ..., m$$

wherein a_{ij} represents the pay-off resulting from the combination of ith event and jth act, while p_i represents the probability of ith event.

To illustrate the use of the principle, we consider Example 13.1 again, the pay-off matrix in respect of which is reproduced in Table 13.4, along with the probabilities of the occurrence of various events as mentioned previously.

TABLE 13.4 Calculation of Expected Pay-Offs

Events E_i	Probability p_i	Act, A_j					
		$A_1: 18$	$A_2: 19$	$A_3: 20$	$A_4: 21$	$A_5: 22$	$A_6: 23$
$E_1: 18$	0.05	360	310	260	210	160	110
$E_2: 19$	0.10	360	380	330	280	230	180
$E_3: 20$	0.30	360	380	400	350	300	250
$E_4: 21$	0.40	360	380	400	420	370	320
$E_5: 22$	0.10	360	380	400	420	440	390
$E_6: 23$	0.05	360	380	400	420	440	460
Expected Pay-off		360	376.5	386	374.5	335	288.5

The calculation of the expected pay-off values, *EP*, for some of the acts is shown here:

For A_1, $EP_1 = 0.05 \times 360 + 0.10 \times 360 + 0.30 \times 360 + 0.40 \times 360 + 0.10 \times 360 + 0.05 \times 360 =$ Rs 360

For A_2, $EP_2 = 0.05 \times 310 + 0.10 \times 380 + 0.30 \times 380 + 0.40 \times 380 + 0.10 \times 380 + 0.05 \times 380 =$ Rs 376.5

For A_6, $EP_6 = 0.05 \times 110 + 0.10 \times 180 + 0.30 \times 250 + 0.40 \times 320 + 0.10 \times 390 + 0.05 \times 460 =$ Rs 288.5

Since the maximum expected pay-off is associated with strategy A_3, the best course of action is to buy 20 copies of the book.

Expected Opportunity Loss or Expected Regret The expected opportunity loss or expected regret criterion is another basis on which a decision may be taken. As we shall observe, this criterion leads to the same conclusion as the expected pay-off criterion.

For our example, the conditional opportunity loss, or regret, matrix along with the probability distribution of demand are reproduced in Table 13.5. As observed already, for any given event, the conditional regret of the optimal act is zero while the conditional regret in respect of any act other than this is positive and equals the difference of the pay-offs of the optimal act and the act adopted.

TABLE 13.5 Calculation of Expected Regret

Events E_i	Probability p_i	Act, A_j					
		$A_1: 18$	$A_2: 19$	$A_3: 20$	$A_4: 21$	$A_5: 22$	$A_6: 23$
$E_1: 18$	0.05	0	50	100	150	200	250
$E_2: 19$	0.10	20	0	50	100	150	200
$E_3: 20$	0.30	40	20	0	50	100	150
$E_4: 21$	0.40	60	40	20	0	50	100
$E_5: 22$	0.10	80	60	40	20	0	50
$E_6: 23$	0.05	100	80	60	40	20	0
Expected Regret		51	34.5	25	36.5	76	122.5

The expected regret for any strategy is determined by summing up the products of the regret values and their respective probabilities. For example, for A_1, we have expected regret,

$$ER_1 = 0.05 \times 0 + 0.10 \times 20 + 0.30 \times 40 + 0.40 \times 60 + 0.10 \times 80 + 0.05 \times 100 = 51.$$

Obviously, under this criterion, the optimal strategy is the one which minimises the expected regret. Since the minimum value occurs at A_3 in the example under consideration, it represents the optimal decision, the same as under the expectation principle.

Expected pay-off of perfect information (EPPI) This is calculated as follows. When the bookstore manager knows that next year's demand is 18 copies, he adopts the strategy of ordering 18 copies of the book and obtains a profit of Rs 360. This occurs 5 per cent of the time since the probability of a demand of 18 copies is known as 0.05. Therefore, the expected profit is $0.05 \times 360 =$ Rs 18. When the manager knows that next year's demand is 19 copies, he orders for an equal number of copies for a profit of Rs 380, and this happens 10 per cent of the time. For this, the expected pay-off equals $0.10 \times 380 =$ Rs 38. Similarly, the expected pay-off for each level of demand can be obtained which, when aggregated, yields the EPPI, as shown here.

$$EPPI = 0.05 \times 360 + 0.10 \times 380 + 0.30 \times 400 + 0.40 \times 420 + 0.10 \times 440 + 0.05 \times 460 = \text{Rs } 411$$

Thus, if the manager can, in any way, know perfectly the demand for the book in advance, and order for as many copies, the store can average a net profit equal to Rs 411. This represents the highest profit that the store can make when perfect information is available. Now, suppose that an agency undertakes to supply this information. We may ask the question as to how much should the store be prepared to pay to the agency for it. In other words, what is the worth of this information? Since the bookstore can make an expected profit of Rs 386 when no such information is available, and an expected profit of Rs 411 when the information is available, the worth of this information is $EPPI - EP = 411 - 386 = 25$. The bookstore should, therefore, pay no more than Rs 25 (per year) for obtaining such information. This value is called the *expected value of perfect information*, (*EVPI*), and it equals the expected regret value of the optimal act.

It is interesting to observe that for each course of action, A_j, the expected profit plus expected regret value equals the expected pay-off of perfect information. That is to say, $EP_j + ER_j = EPPI$, for $j = 1, 2, ...$ For example, for A_1, $EP_1 = 360$, $ER_1 = 51$ and $360 + 51 = 411$ or $EPPI$, and for A_2, $EP_2 = 376.5$, $ER_2 = 34.5$ which add up to 411.

Example 13.2 Technico Ltd. has installed a machine costing Rs 4 lacs and is in the process of deciding on an appropriate number of a certain spare parts required for repairs. The spare parts cost Rs 4,000 each but are available only if they are ordered now. In case the machine fails and no spares are available, the cost to the company of mending the plant would be Rs 18,000. The plant has an estimated life of 8 years and the probability distribution of failures during this time, based on experience with similar machines, is as follows:

No. of failures during 8-yearly period :	0	1	2	3	4	5	6+
Probability :	0.1	0.2	0.3	0.2	0.1	0.1	0

Ignoring any discounting for time value of money, determine the following:

(a) The optimal number of units of the spare part on the basis of (i) minimax principle, (ii) minimin principle, (iii) Laplace principle, (iv) Hurwicz principle (taking $a = 0.7$), and (v) expected cost principle.

(b) The expected number of failures in the 8-year period.

(c) The regret table, and the optimal choice on the basis of least expected regret criterion.

(d) EVPI.

If we let F represent the number of failures, S the number of spares, and C the total cost, the cost function can be stated as follows:

$$C = 4{,}000S, \quad \text{when } F \leq S$$
$$= 4{,}000S + 18{,}000\,(F - S) \quad \text{when } F > S$$

On the basis of this information, the cost matrix has been constructed and shown in Table 13.6. Also shown in the table are probabilities of the various numbers of failures.

TABLE 13.6 Cost Matrix

(All cost figures in '000 Rs)

No. of Failures E_i	Probability p_i	No. of Spares, A_j					
		$A_1: 0$	$A_2: 1$	$A_3: 2$	$A_4: 3$	$A_5: 4$	$A_6: 5$
$E_1: 0$	0.1	0	4	8	12	16	20
$E_2: 1$	0.2	18	4	8	12	16	20
$E_3: 2$	0.3	36	22	8	12	16	20
$E_4: 3$	0.2	54	40	26	12	16	20
$E_5: 4$	0.1	72	58	44	30	16	20
$E_6: 5$	0.1	90	76	62	48	34	20
Column Minima		0	4	8	12	16	20
Column Maxima		90	76	62	48	34	20
Simple Average Cost		45	34	26	21	19	20
Expected Cost		41.4	29.2	20.6	17.4	17.8	20.0

(a) No. of units on the basis of different principles:

(i) *Minimax* The maximum values in each of the columns are indicated by the *column maxima* row. The minimum of these being 20, the decision on the basis of the minimax rule would be to buy 5 spare parts.

(ii) *Minimin* From the minimum values contained in the row entitled *column minima*, it may be observed that the least value is equal to zero. Thus, the optimal number of spare parts is nil.

(iii) *Laplace Principle* According to this principle, the different events, E_i's, are assumed to be equally probable. Thus, the decision is taken on the basis of the simple average cost values (determined without using the given probability values). Since the simple average cost is the minimum for A_5, it follows that the optimal number of spares, according to the Laplace rule, is 4.

(iv) *Hurwicz Principle* We shall first calculate the Hurwicz Criterion for each of the strategies. In the context of cost data, Hurwicz Criterion, $HC = \alpha$ (Min Value) + $(1 - \alpha)$ (Max Value). Its value for various strategies is as follows:

For A_1	:	$0.7 \times 0 + 0.3 \times 90 = 27.0$
For A_2	:	$0.7 \times 4 + 0.3 \times 76 = 25.6$
For A_3	:	$0.7 \times 8 + 0.3 \times 62 = 24.2$
For A_4	:	$0.7 \times 12 + 0.3 \times 48 = 22.8$
For A_5	:	$0.7 \times 16 + 0.3 \times 34 = 21.4$
For A_6	:	$0.7 \times 20 + 0.3 \times 20 = 20.0$

Being *lowest* for A_6, optimal strategy is to keep 5 spare parts.

(v) *Expected Cost Principle* From Table 13.6, we observe that the minimum value appears against the strategy A_4. Thus, according to the expectation principle, the optimal policy is to store 3 spare parts, the expected cost being Rs 17.4 thousand.

(b) Expected number of failures in the 8-year period,

$$E(F) = \sum_{i=1}^{6} p_i E_i$$

Thus, $E(F) = 0.1 \times 0 + 0.2 \times 1 + 0.3 \times 2 + 0.2 \times 3 + 0.1 \times 4 + 0.1 \times 5 = 2.3$

(c) The regret table can be derived from Table 13.6. For it, the *least* cost value in each row would be subtracted from other values. The differences represent the regret values. Table 13.7 depicts these.

TABLE 13.7 Regret Table

No. of Failures E_i	Probability p_i	No. of Spares, A_j					
		$A_1:0$	$A_2:1$	$A_3:2$	$A_4:3$	$A_5:4$	$A_6:5$
$E_1:0$	0.1	0	4	8	12	16	20
$E_2:1$	0.2	14	0	4	8	12	16
$E_3:2$	0.3	28	14	0	4	8	12
$E_4:3$	0.2	42	28	14	0	4	8
$E_5:4$	0.1	56	42	28	14	0	4
$E_6:5$	0.1	70	56	42	28	14	0
Expected Regret		32.2	20	11.4	8.2	8.6	10.8

Since the expected regret for strategy A_4 is the least, the optimal policy according to this criterion, as in case of the expected cost principle, is to buy 3 spare parts.

(d) The expected value of perfect information, EVPI, when the pay-off matrix indicates cost, is defined as follows:

EVPI = Expected cost with optimal policy – Expected cost with perfect information

For our example, the expected cost with the optimal policy is 17.4 thousand rupees while the expected cost with perfect information is 9.2 thousand rupees, as follows:

Event	Cost	Prob.	Prob. × Cost
$E_1:0$	0	0.1	0.0
$E_2:1$	4	0.2	0.8
$E_3:2$	8	0.3	2.4
$E_4:3$	12	0.2	2.4
$E_5:4$	16	0.1	1.6
$E_6:5$	20	0.1	2.0
		Expected Cost	9.2

Thus, EVPI = 17.4 – 9.2 = 8.2 thousand rupees.

13.2.3 Bayesian Decision Rule: Posterior Analysis

In the preceding analysis of decision-making under risk, we have seen how probability information about the various states of nature is used for determining the expected pay-off values resulting from different courses of action. The Bayesian decision rule represents an extension of this. In this approach, the optimal strategy is chosen using the expected value criterion while the expected pay-offs are calculated by using *posterior probabilities*.

In using the Bayesian rule, the preliminary or prior information (from the past experience) of the decision-maker is revised on the basis of some additional information about the states of nature: the prior probabilities are converted into the posterior probabilities using the information. The use of these posterior probabilities for taking the decisions is likely to enable better decisions. In a given situation, the new information may be obtained through test research, raw material sample testing, etc.

We shall illustrate the use of this rule by means of the following example.

Example 13.3 Suppose Delhi Developers Limited (DDL), a construction company, has recently acquired a piece of land in a city on which it plans to construct a shopping complex. The company has now to decide the size of the complex. It is considering three options: a small-sized complex with 40 condominiums and a multiplex, say A_1; a medium-sized complex consisting of 60 condominiums and a multiplex, call it A_2; and a large-sized complex with 100 condominiums, say A_3. The company feels that the overall demand for the condominiums built would be either high or low. The returns from the project will obviously depend on what size of complex is developed and what the level of demand eventually turns out to be. The pay-offs (in thousands of rupees) expected under various event-action combinations, together with estimated probabilities of the likely demand, are given here:

		Act, A_j		
Event	*Probability*	*A_1: Small complex*	*A_2: Medium complex*	*A_3: Large complex*
High demand	0.4	1,800	2,200	4,200
Low demand	0.6	1,000	600	–1,200

Further, this company has the choice of engaging a marketing research firm to conduct a survey for it so that it can take a 'more informed' decision. Suppose the outcomes of the marketing research study may be indicated as I_1 and I_2 as follows:

I_1: A favourable report, indicating high demand for condominiums

I_2: An unfavourable report, indicating a low demand for condominiums

The company knows that, in all likelihood, the information provided by the marketing research firm would not always be cent per cent correct. Suppose that past record of the research firm on similar studies has led to the following estimates of the relevent probabilities:

	Marketing Research Report	
Event	*Favourable (I_1)*	*Unfavourable (I_2)*
E_1: High demand	0.9	0.1
E_2: Low demand	0.2	0.8

The marketing research firm has asked for Rs 300,000 as the fee for undertaking the study. How should the construction company proceed?

Here the company has two options to decide on the size of the shopping complex: one, on the basis of only its own estimates of the likely demand, and two, engaging and taking into consideration the report of the marketing research firm. The evaluation of the former is the same as the analysis that we have considered so far. It is also called *prior analysis* because it uses only prior probabilities. On the other hand, the latter involves incorporating the information obtained through marketing research into the analysis and take decision thereafter. This is termed as the *posterior analysis*. We consider them in turn now.

Prior analysis The pay-off table for the problem is reproduced in Table 13.8 and the expected values for the various acts are shown calculated.

TABLE 13.8 Calculation of Expected Pay-off

Event	*Probability*	*Act, A_j*		
		A_1: Small complex	*A_2: Medium complex*	*A_3: Large complex*
E_i	p_i			
E_1: High demand	0.4	1,800	2,200	4,200
E_2: Low demand	0.6	1,000	600	–1,200
Expected Pay-off		1,320	1,240	960

Here the expected pay-off for action A_1 is the maximum; it represents the optimal course of action. With optimal policy of A_1, EP_1 = Rs 1,320 thousand. Thus, according to the given probabilities of high and low demands, the company should construct a small-sized shopping complex.

It is evident from Table 13.8 that in the event of high demand for condominiums, the best strategy is a large-sized complex with a pay-off of Rs 4,200 thousand and, in case of low demand, a small-sized complex involving a pay-off of Rs 1,000 thousand is the best course. Using these values, we can determine the expected pay-off of perfect information, EPPI, for this decision problem as shown here:

Event	*Probability*	*Pay-off ('000 Rs)*	*Pay-off × Probability*
E_1: High demand	0.4	4,200	1,680
E_2: Low demand	0.6	1,000	600
		EPPI =	2,280

Thus, under conditions of complete and certain information, the expected profit would be Rs 2,280 thousand. From this, the expected value of perfect information, EVPI, can be obtained as follows:

$$EVPI = EPPI - EP = 2{,}280 - 1{,}320 = \text{Rs } 960 \text{ thousand}$$

Posterior analysis We now consider what decision the company would take if it engages the marketing research firm. The additional information provided by the firm may call for a revision of the decision of small-sized complex earlier or reinforce it. The approach involves revising the prior probabilitites of high and low demand to calculate posterior probabilities in the first place, and then using these posterior probabilities to reach a decision. Accordingly, this is called *posterior analysis*.

Let us consider how posterior probabilities may be calculated and used. In the first place, it may be noted that the probabilities given in the context of marketing firm's past records are in fact conditional probabilities. To illustrate, 0.90 is the answer to the following question: given that the state of nature eventually turns out to be a high demand, what is the probability that the marketing research study shall give a favourable report? This is, in other words, the conditional probability $P(I_1/E_1)$. Similarly, $P(I_1/E_2) = 0.20$, $P(I_2/E_1) = 0.10$ and $P(I_2/E_2) = 0.80$ can be appropriately interpreted. Obviously, on the basis of the probability estimates, we can place a high degree of confidence on the market research report. This is because when the true state of nature is high level of demand for the condominiums, E_1, the report shall be favourable 90 per cent of time, and unfavourable the 10 per cent. Similarly, when the true state is low demand, there is an 80 per cent likelihood of the report also being an unfavourable one, giving correct information.

Using the prior and conditional probabilities, we determine the total probability that the research report will be favourable, and the total probability that the research report will be unfavaourable. Since both, the favourable and unfavourable report may be associated with the events of high and low demands for condominiums, we can determine these probabilities as follows:

$$P(I_1) = P(E_1 \cap I_1) + P(E_2 \cap I_1)$$
$$= P(E_1) \times P(I_1/E_1) + P(E_2) \times P(I_1/E_2)$$

Substituting all values, we get

$$P(I_1) = 0.4 \times 0.9 + 0.6 \times 0.20$$
$$= 0.36 + 0.12 = 0.48$$

Similarly,

$$P(I_2) = P(E_1 \cap I_2) + P(E_2 \cap I_2)$$
$$= P(E_1) \times P(I_2/E_1) + P(E_2) \times P(I_2/E_2)$$
$$= 0.4 \times 0.1 + 0.6 \times 0.80 = 0.52$$

Now, we calculate the posterior probabilities for the events E_1 and E_2 under the conditions (a) that a favourable report is given; and (b) an unfavourable report is given by the marketing research firm. These probabilities would then be used to determine optimal course of action under each of the two situations represented by I_1 and I_2.

(a) When a Favourable Report is Given (Indicator I_1)

The posterior probabilities for the events E_1 and E_2, when a favourable report is given by the marketing research firm, are shown calculated in Table 13.9.

TABLE 13.9 Calculation of Posterior Probabilities

Event E_i	*Prior Probability,* $P(E_i)$	*Conditional Probability,* $P(I_1/E_i)$	*Joint Probability,* $P(I_1 \cap E_i)$	*Posterior Probability,* $P(E_i/I_1)$
E_1	0.4	0.90	0.36	0.36/0.48 = 0.75
E_2	0.6	0.20	0.12	0.12/0.48 = 0.25

The posterior probabilities, calculated by using Bayes' theorem, suggest that if marketing research report was in fact a favourable one, then the probability of the event of high demand, E_1 would be 0.75 and that of the event of low demand, E_2, would be 0.25. We shall use these probabilities instead of 0.4 and 0.6 respectively to determine the optimal course of action. The posterior probabilities of the event, the pay-off matrix and the expected values using these probabilities are contained in Table 13.10.

TABLE 13.10 Determination of Optimal Course of Action

Event E_i	Probability p_i	Act, A_j: A_1: Small complex	A_2: Medium complex	A_3: Large complex
E_1 : High demand	0.75	1,800	2,200	4,200
E_2 : Low demand	0.25	1,000	600	−1,200
Expected Pay-off		1,600	1,800	2,850

Since the expected pay-off for the act A_3 is the highest, we conclude that given that a favaourable research report is obtained, the best course of action would be the construction of a large-sized shopping complex. This has expected pay-off of Rs 2,850 thousand.

(b) When an Unfavourable Report is Given (Indicator I_2)

Posterior probabilities in the event of indicator I_2–unfavourable research report–are calculated and shown in Table 13.11.

TABLE 13.11 Calculation of Posterior Probabilities

Event E_i	Prior Probability, $P(E_i)$	Conditional Probability, $P(I_2/E_i)$	Joint Probability, $P(I_2 \cap E_i)$	Posterior Probability, $P(E_i/I_2)$
E_1	0.4	0.10	0.04	0.04/0.52 = 0.0769
E_2	0.6	0.80	0.48	0.48/0.52 = 0.9231
		Total	0.52	

From this table, it is clear that if an unfavourable report is obtained, then the probabilities of high and low demand for condominiums would stand revised at 0.0769 and 0.9231 respectively. Using these probabilities we can calculate the expected pay-offs of various courses of action and choose the appropriate one. The pay-off values and the expected pay-offs using the posterior probabilities are given in Table 13.12.

TABLE 13.12 Determination of Optimal Course of Action

Event E_i	Probability p_i	Act, A_j: A_1: Small complex	A_2: Medium complex	A_3: Large complex
E_1: High demand	0.0769	1,800	2,200	4,200
E_2: Low demand	0.9231	1,000	600	−1,200
Expected Pay-off		1,061.52	723.04	−784.74

Since the expected pay-off of the act A_1 is the largest, the company would decide to construct a small-sized shopping complex in case an unfavourable report indicating low demand is given by the marketing research firm.

Thus, when a favourable market research report is obtained, the optimal course of action would be A_3 with an expected pay-off of Rs 2,850 thousand, and in the event of an unfavourable report, the optimal act would be A_1, with an associated expected pay-off equal to Rs 1,061.52 thousand. But these decisions are conditional in the sense that either of them can be taken only when the nature of report is known. Since there is a likelihood of 0.48 for the report to be favourable, and 0.52 for it to be otherwise, we can get the expected pay-off value as shown in Table 13.13.

TABLE 13.13 Calculation of Expected Pay-off

Indicator	*Probability*	*Conditional Pay-off of the Best Act*	*Expected Value*
I_1: Favourable Report	0.48	2,850	1,368
I_2: Unfavourable Report	0.52	1,061.52	552
Expected Pay-off			1,920

To conclude, the expected pay-off of the optimal decision is Rs 1,920 thousand, if it is based on the market research information, as against Rs 1,320 thousand without such information.

Expected Value of Sample Information, EVSI As discussed earlier, the company in question would expectedly do best to build a large-sized shopping complex when a favourable research report is obtained, and a small-sized complex when the report is not favourable. In view of the fact that the company would have to incur extra cost by payment to the marketing research firm, it would like to know the value to be placed on the information provided by the research firm. In general terms, the information on the basis of which prior probabilities are revised is called the *sample information*. Thus, we have to determine here the expected value of sample information, EVSI. It is defined as follows:

EVSI = Expected pay-off *with* sample information *minus* Expected pay-off *without* sample information

As we have already calculated, the expected pay-off with sample information is Rs 1,920 thousand while the expected pay-off when no sample information is called for (i.e. without calling for market research report) is Rs 1,320 thousand. Therefore, EVSI = 1,920 – 1,320 = Rs 600 thousand. The company can, therefore, pay an amount up to Rs 600,000 to the marketing research firm for the research report. Since it is given here that the marketing research firm has asked a fee of Rs 300,000, it is worth engaging it.

Efficiency of EVSI

We have already determined that the expected value of perfect information, EVPI, for the given problem is Rs 960 thousand. In case the market research would always yield absolutely correct information, its value would match this amount, and thus it would be cent per cent efficient. However, as the report is not always likely to yield correct information, we can measure its efficiency by relating EVSI to EVPI. Thus, an efficiency index, *EI*, of smaple information can be obtained as follows:

$$EI = \frac{\text{EVSI}}{\text{EVPI}} \times 100$$

In our example,

$$EI = \frac{600}{960} \times 100 = 62.5\%$$

Thus, the sample information is about 63 per cent as efficient as perfect information.

Calculation of efficiency index enables the decision-maker to decide whether or not to seek the information and compare various sources through which the needed information may be obtained.

13.3 MULTI-STAGE DECISION-MAKING PROBLEMS: DECISION TREE

In the discussion of decision problems uptil now, our concern has been with the single stage problems wherein the decision-maker has to select the best course of action on the basis of whatever information is achievable at a point in time. We shall now consider the decision situations that involve multiple stages. Also called the *sequential decision* problems, they are characterised by a sequence of decisions in which following each decision, a chance event occurs which in turn influences the next decision.

In analysing multiple stage decision situations, we have to evaluate the decision proceeding in a backward manner by evaluating the best course of action at the later stages to decide the best action at the earlier stages. For this purpose, the *decision tree* or the *decision flow diagram* as it is sometimes called, is a very effective device.

A decision tree is a graphic representation of the sequences of action-event combinations available to the decision-maker. It depicts in a systematic manner all possibie sequences of decisions and consequences. Each such sequence is shown by a distinct path through the tree. A decision tree enables the decision-maker to see the various elements of his problem in proper perspective and in a systematic manner. It may be mentioned that the criterion on the basis of which the decisions are made in the decision tree approach is generally the expectation principle. Thus, we may choose the alternative that maximises the expected profit, or the alternative that minimises the expected cost . . . and so on.

Let us consider the decision tree analysis with the help of the following example.

Example 13.4 An oil company has recently acquired rights in a certain area to conduct surveys and test drillings to lead to lifting oil if it is found in commercially exploitable quantities.

The area is considered to have good potential for finding oil in commercial quantities. At the outset, the company has the choice to conduct further geological tests or to carry out a drilling programme immediately. On the known conditions, the company estimates that there is a 70 : 30 chance of further tests showing a 'success'.

Whether the tests show the possibility of ultimate success or not or even if no tests are undertaken at all, the company could still pursue its drilling programme or alternatively consider selling its rights to drill in the area. Thereafter, however, if it carries out the drilling programme, the likelihood of final success or failure is considered dependent on the foregoing stages. Thus:

- if 'successful' tests have been carried out, the expectation of success in drilling is given as 80 : 20.
- if the tests indicate 'failure', then the expectation of success in drilling is given as 20 : 80.
- if no tests have been carried out at all, the expectation of success in drilling is given as 55 : 45.

Costs and revenues have been estimated for all possible outcomes and the net present value of each is as follows:

Outcome		Net Present Value (Rs million)
Success:	With prior tests	100
	Without prior tests	120
Failure:	With prior tests	– 50
	Without prior tests	– 40
Sale of exploitation rights:	Prior tests show 'success'	65
	Prior test show 'failure'	15
	Without prior tests	45

(a) Draw up a decision (probability) tree diagram to represent the above information; and

(b) Evaluate the tree in order to advise the management of the company on its best course of action.

(MBA, Delhi, 2007)

The decision tree corresponding to the given problem is depicted in Figure 13.1.

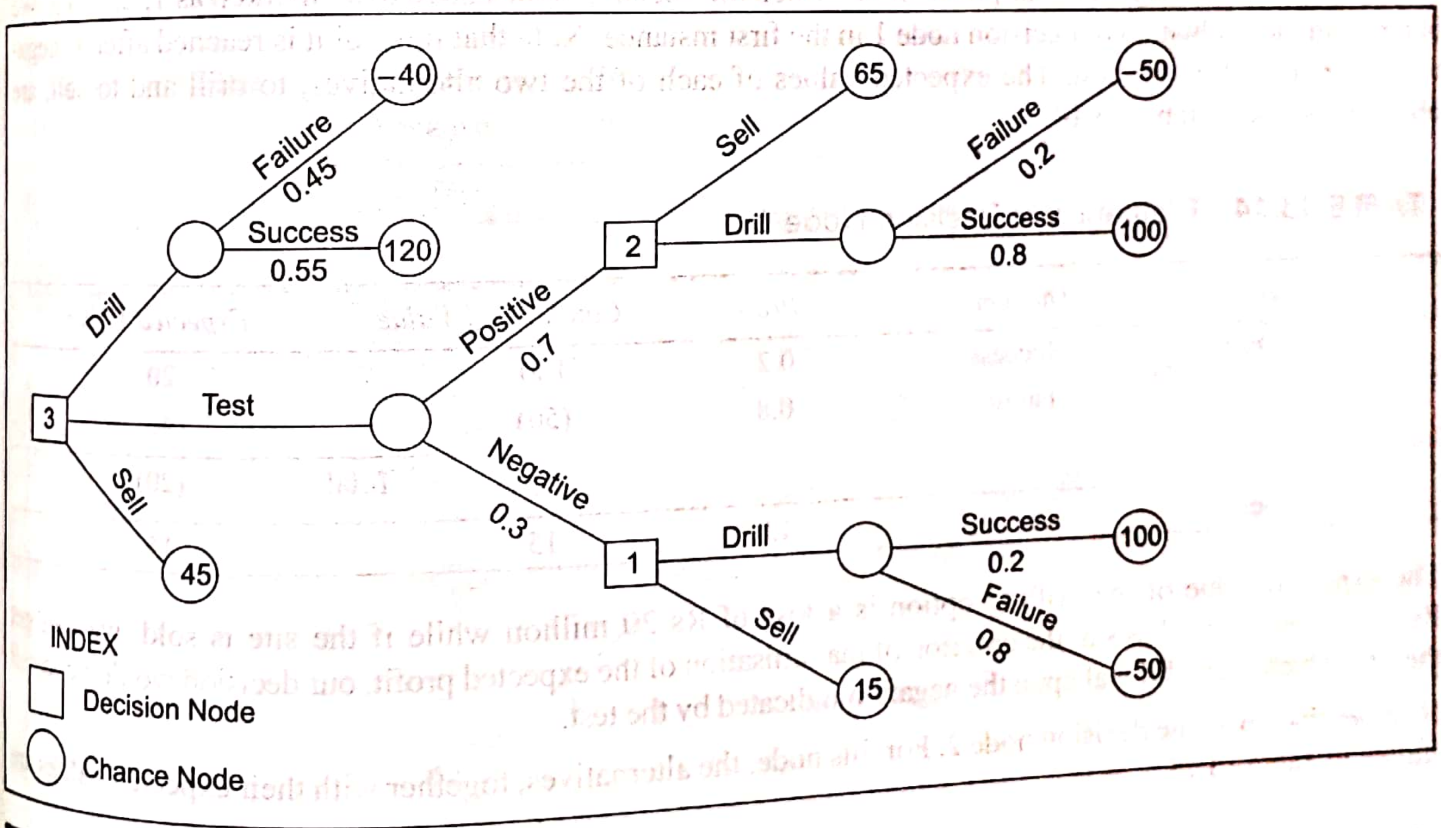


Figure 13.1 *Decision Tree*

Observe that the tree has several branches which originate from squares or from circles. A square represents a *decision node* or *decision fork* at which the decision-maker has to take a decision, while a circle represents a *chance node* or *chance fork* at which events (i.e. the states of nature) are branched out. At decision node marked 3, there are branches, representing three alternatives of drilling, testing for oil, and selling of rights, of which the decision-maker has to select one. Now, if the company decides to drill, there are two possible events—it may get oil or not, which are shown as branches emanating from circle, whose probabilities are 0.55 and 0.45, respectively. A profit of Rs 120 million would result in case oil is obtained and a loss of Rs 40 million if it is not.

The second alternative at the decision node 3 is to go for a test, which may give positive or negative results, with probabilities 0.7 and 0.3, respectively. In case a positive result is indicated, a choice has to be made as to whether to sell the rights for Rs 65 million or to drill, which is likely to succeed and fail with chances 80 : 20. Therefore, a decision node number 2 is shown by a square. Similarly, for a negative indication, there are two options—to sell for Rs 15 million or to drill, which has a 20 per cent chance of success. Thus, decision node 1 is indicated here.

The third alternative is to sell the rights for Rs 45 million.

The Method of Solution As mentioned earlier, the decisions have to be evaluated in a backward manner by evaluating the best course of action at the later stages so as to decide on the best course of action on the earlier stages. Thus, a solution to the problem is obtained by working backwards, from right to left, through the tree. This is called the *rollback* technique. In this technique, we assume that we have reached various decision nodes on the tree and then decide on the optimal act conditional on having reached the node being analysed. Thus, on reaching any decision node, we evaluate each of the alternative courses of action available there and select the most appropriate one. We begin with the rightmost decision node and, after having analysed it, we move backward and analyse the preceding decision node in a similar manner. The process is continued till the first, the leftmost, decision node is analysed. The decision at the first node represents the best initial decision.

Let us consider the solution to our problem for which the decision nodes have been marked as 1, 2, and 3. We begin with the evaluation of decision node 1 in the first instance. Note that this point is reached after a 'negative' is indicated by the test. The expected values of each of the two alternatives, to drill and to sell, are obtained below in Table 13.14.

TABLE 13.14 Evaluation of Decision Node 1

Alternative		*Outcome*	*Prob.*	*Conditional Value*		*Expected Value*
1.	Drill	Success	0.2	100		20
		Failure	0.8	(50)		(40)
					Total	(20)
2.	Sell		1.0	15		15

The expected value of the drilling option is a loss of Rs 20 million while if the site is sold, we can get Rs 15 million. On the basis of the criterion of maximisation of the expected profit, our decision would be to sell the site, which is conditional upon the negation indicated by the test.

Now we shall evaluate decision node 2. For this node, the alternatives, together with their expected values are shown in Table 13.15.

TABLE 13.15 Evaluation of Decision Node 2

Alternative		*Outcome*	*Prob.*	*Conditional Value*		*Expected Value*
1.	Sell		1.0	65		65
2.	Drill	Success	0.8	100		80
		Failure	0.2	(50)		(10)
					Total	70

The two alternative courses of action, namely selling and drilling, have expected values equal to Rs 65 million and Rs 70 million, respectively. Obviously, therefore, *provided* that a positive result is indicated by the test, the best course would be to go in for oil drilling.

Now, at this stage we have two conditional decisions—sell the site if a 'negative' is obtained on the test and drill in case the test indicates a 'positive'. At each node, the branches on which we have not to move, representing the options ruled out, have been shown cancelled. Next, we move to the decision node 1, where a decision has to be taken whether to drill at the outset, to undertake a test or to sell the rights outright. The alternatives, along with their expected values are shown in Table 13.16. Also, the expected values associated with the different chance nodes and the decision nodes are indicated in Figure 13.2 in which the decision tree given in Figure 13.1 is reproduced.

TABLE 13.16 Evaluation of Decision Node 3

Alternative		*Outcome*	*Prob.*	*Conditional Value*		*Expected Value*
1.	Drill	Success	0.55	120		66
		Failure	0.45	(40)		(18)
					Total	48
2.	Test	Positive	0.7	70		49.0
		Negative	0.3	15		4.5
					Total	53.5
3.	Sell		1.0	45		45

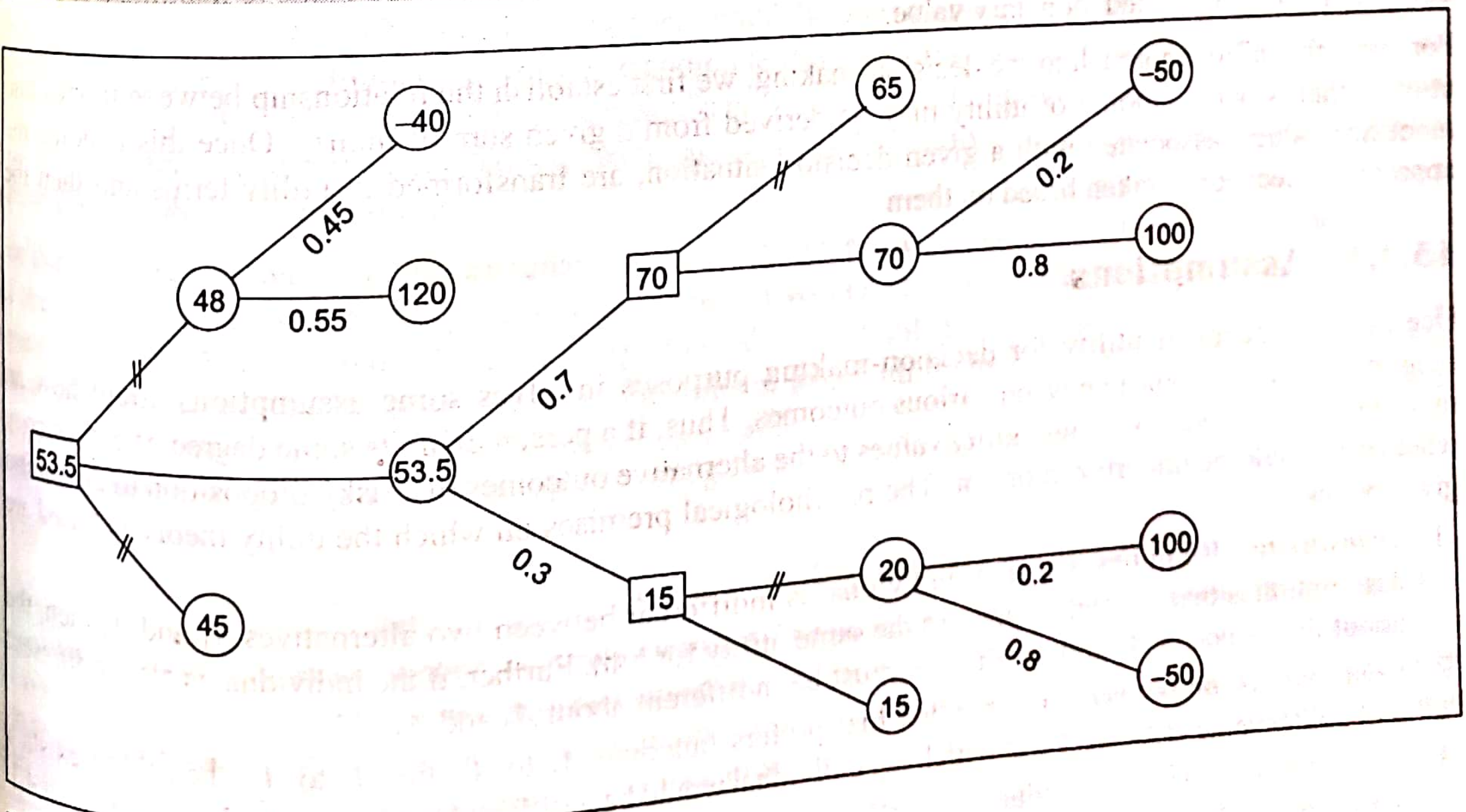


Figure 13.2 *Decision Tree*

As we may observe, the expected value of the alternative of carrying out a test is Rs 53.5 million, which is the highest of the three. Therefore, it is better to test before drilling. This is the initial decision. The overall decision can now be stated as: The test be carried out. If it proves negative, the rights should be sold to give a return of Rs 15 million. To proceed with drilling, if that happens, would expectedly lead to a loss. However, if the test proves positive, the drilling should be undertaken.

13.4 UTILITY THEORY: UTILITY AS BASIS FOR DECISION-MAKING

Suppose that a choice is offered between the following two alternatives: A_1—a certain gift of Rs 10,000, and A_2—a gift of Rs 25,000 if a coin, when tossed, falls head up, and nothing if the coin shows up tail. Clearly, most of us are likely to choose alternative A_1 with a sure amount of Rs 10,000 in preference to a risky A_2, although the expected (monetary) value for A_2, equal to $0.5 \times 25000 + 0.5 \times 0 =$ Rs 12,500, is larger than that of alternative A_1. It may therefore be reasonably argued that people do not always take decisions as will maximise their expected monetary value. Thus, the expectation principle, or the expected monetary value (EMV) criterion, which we have considered so far, does not always provide an adequate and satisfactory basis for decision-making. An alternative criterion that could be used for decision-making, which is consistent with the choice of alternatives like A_1 in our example is the one provided by Von Neumann and Morgenstern. According to them, the decisions are made so as to maximise *expected utility* rather than expected monetary value. In preferring A_1 over A_2, they contend, the decision-maker derives greater utility from alternative A_1 in comparison with the utility he would derive from alternative A_2. If he was indifferent between the two alternatives, the contention is that his expected utility is same for both the alternatives. It is possible to make generalisation about a person's utility function for a commodity, money in the present context, which are consistent logically and with observation of repeated decisions. Thus, it may be reasonably taken that decisions are made to maximise expected utility and not the expected monetary value.

For using the utility approach to the decision-making, we first establish the relationship between money and utility—that is, what amount of utility may be derived from a given sum of money. Once this is done, the monetary values, associated with a given decision situation, are transformed in utility terms and then the appropriate decision is taken based on them.

13.4.1 Assumptions

Use of the concept of utility for decision-making purposes involves some assumptions about how an individual reacts to choice between various outcomes. Thus, if a person exhibits some degree of consistency in his preferences, we can assign utility values to the alternative outcomes of a risky proposition to determine whether it would be undertaken or not. The psychological premises on which the utility theory is based are given below.

1. *Transitivity* It implies that if an individual is indifferent between two alternatives A_1 and A_2 then, the assumption is that, A_1 and A_2 possess the same utility for him. Further, if the individual is also indifferent about the choice of A_1 and A_3, then he must be indifferent about A_2 and A_3.
2. *Continuity of preference* If an individual prefers outcome A_1 to A_2 and A_2 to A_3, then there exists a probability value $\alpha, 0 < \alpha < 1$, at which the individual would be indifferent between A_2 and $[\alpha A_1 + (1-\alpha)A_3]$.
3. *Independence* If an individual is indifferent between alternatives A_1 and A_2, and he is also indifferent between A_3 and A_4, then for any probability $\alpha\,(0 < \alpha < 1)$ he would be indifferent between $\alpha A_1 + (1-\alpha)A_3$ and $\alpha A_2 + (1-\alpha)A_4$.

4. *Desire for higher probability of success* According to this, if an individual prefers A_1 to A_2 and there are two probabilities α_1 and α_2 such that $\alpha_1 > \alpha_2$, then he would prefer $\alpha_1 A_1 + (1-\alpha_1)A_2$ to $\alpha_2 A_1 + (1-\alpha_2)A_2$. Also, for two alternatives with identical rewards, the one with the higher probability of success would be desired by the decision-maker.

5. *Compound probability* If A_1 and A_2 are two alternatives and α_1, α_2 and α_3 are three probabilities, an individual will be indifferent between $\alpha_1[\alpha_2 A_1 + (1-\alpha_2)A_2] + (1-\alpha_1)[\alpha_3 A_1 + (1-\alpha_3)A_2]$ and $\alpha_1\alpha_2 + (1-\alpha_1)\alpha_3] (A_1) + [1 - \alpha_1\alpha_2 + (1-\alpha_1)\alpha_3)] (A_2)$.

13.4.2 Utility Measures and Utility Function

With the assumptions of utility theory, we now proceed to discuss how utility can be measured.

Von Neumann and Morgenstern have proposed an index for measurement of utility, which is a special type of cardinal measure. It measures utility in situations that involve risk for the decision-maker. This utility index is designed for predictive purposes, and allows to predict which of several bets a person would prefer and thus enables him to take decisions.

As stated earlier, the theory of utility postulates that a rational decision-maker will always decide to maximise utility or expected utility. The expected utility of a risky alternative is defined as the aggregate of the products of the utility values of all its possible outcomes and their respective probabilities. For an alternative *A*, if there are two possible outcomes x_1 and x_2 with respective probabilities of α and $1-\alpha$, and that respective utilities are Ux_1 and Ux_2, we have,

Expected utility of A,

$$EU_A = \alpha Ux_1 + (1-\alpha)Ux_2$$

The Utility Function Before we consider how the utility function, relating utility and money for a decision-maker, may actually be derived, it may be mentioned that the utility function should possess the property of *completeness*. A utility function is said to be complete if it measures the utility for all the possible alternatives that are available. Thus, whatever outcomes of a proposition or set of propositions are likely, it should be possible to obtain the utility associated with each one of them. If a utility function is complete, it allows for a comparison of the different outcomes.

For deriving a utility function, first we select two points of reference and assign arbitrary utility values to both of them. The largest and the smallest monetary values involved in a given situation may be taken to be the reference points. Then we may arbitrarily assign the values of 0 utils to the smallest value and 10 utils to the other one—the *utils* being the utility index or the units in which utility may be measured and expressed. For instance, if the lowest monetary value in a decision situation is – Rs 10,000 and the highest one is Rs 50,000, we shall assign a utility of zero utils to – Rs 10,000 and of 10 utils to Rs 50,000.

Thus,

$$U_{-10,000} = 0; \quad \text{and} \quad U_{50,000} = 10$$

This choice of 0 and 10, as mentioned, is purely arbitrary. The numbers chosen could as will be 20 and 752, for example. The utility scale in this sense may be compared with the scale of measuring temperature, where centigrade and fahrenheit, both the scales measure temperature but give different readings for the freezing and boiling points of water. The only point to remember in selecting utility values for the two points is that the greater monetary value should be assigned a larger number and the smaller value be assigned a smaller one. It is based on the assumption that money, like any other commodity, is desirable and people prefer more of it to less of it, and, therefore, higher amounts of money have greater utility than the smaller ones.

Continuing with the above situation where $U_{-10,000} = 0$ and $U_{50,000} = 10$, suppose that the decision-maker is asked the question that if he is given a choice between a cash-certain award of Rs 20,000 on the one hand, and bet involving a loss of Rs 10,000 or a gain of Rs 50,000, occurring with equal probabilities on the other, which one would he prefer? If he opts for the first alternative, it is interpreted that according to his preference, the utility of a sum of Rs 20,000 is greater than the expected utility of the bet which equals $0.5\ U_{-10,000} + 0.5\ U_{50,000} = 0.5 \times 0 + 0.5 \times 10 = 5$ units. For him, then $U_{20,000} > 5$ utils. Next, if he is given a choice between a sure amount of Rs 10,000 and the bet as before (that is, the amount – Rs 10,000 and Rs 50,000 with equal probabilities), and if he prefers the second alternative, then it would be similarly concluded that for him $U_{10,000}$ is less than the expected utility of the bet, 5 utils. In a similar way, if we keep on lowering the amount from Rs 20,000, or increasing from Rs 10,000, and asking for the preferences of the decision-maker as before, a value shall be obtained at which he would be indifferent between the two. For him, if this amount equals Rs 16,000, then we say that the utility of Rs 16,000 to that individual equals the expected utility of the given bet. Thus,

$$U_{16,000} = 0.5\ U_{-10,000} + 0.5\ U_{50,000} = 5 \text{ utils.}$$

Next we vary the probabilities in the bet. If the decision-maker is offered a bet involving a loss of Rs 10,000 with a probability, say 1/10 and a gain with a probability 9/10, or, alternatively, a gift of Rs 40,000, and if he prefers the sure award of Rs 40,000 then it is reckoned that the utility of this amount is greater than the expected utility of the bet. Then we keep on lowering down the amount of sure gift to arrive at a value for which he would be indifferent. If this amount is Rs 36,000, we have

$$U_{36,000} = 0.1\ U_{-10,000} + 0.9\ U_{50,000}$$
$$= 0.1 \times 0 + 0.9 \times 10 = 9 \text{ utils}$$

In a similar manner, the different probability values for the reference points may be taken and the particular amounts at which the decision-maker would be indifferent to each of the values determined. The utility values for each of the amounts are then obtained in the same manner as discussed above.

Once the utility values for several amounts of money are derived, they are plotted on the graph as shown in Figure 13.3.

The amounts of money are shown on the *X*-axis while the utility measure is depicted vertically. The pairs of money amount and corresponding utility value are plotted on the graph and then they are joined by a *continuous curve*. This is the utility function.

Notice from this function that $U_{16000} = 5$ utils, and $(4/9)U_{-10000} + (5/9)U_{36000} = (4/9)0 + (5/9)9 = 5$ utils. As a check on the utility function derived, the decision-maker should be indifferent between a sure amount of Rs 16,000 and a bet involving a loss of Rs 10,000 with a probability 4/9 and a gain of Rs 36,000 with a probability equal to 5/9. If he is not, it implies that the individual in question is not exhibiting consistency in his preferences and the assumptions stated previously are not being met with, and so the utility function needs revision. In case the individual is consistent, the utility function can be used for decision-making.

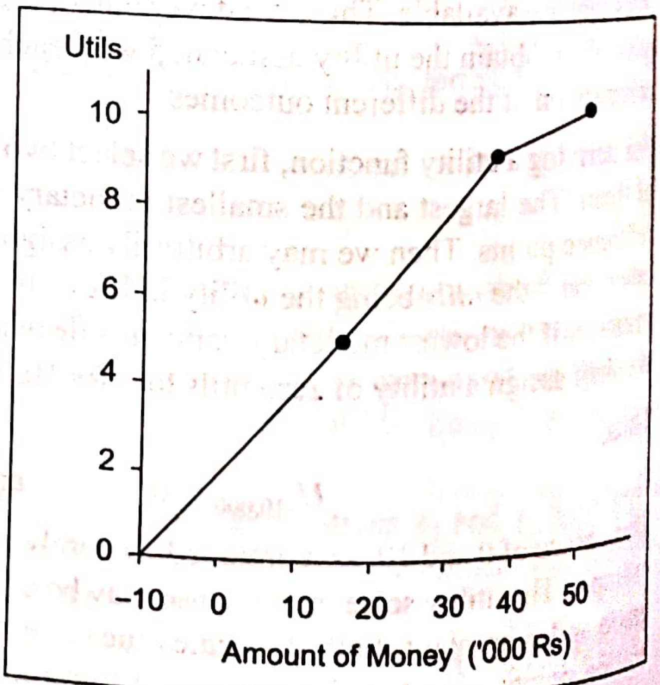


Figure 13.3 *Utility Function*

Before we discuss the use of utility function for decision-making purposes, some observations on the shape of utility functions follow. As stated already, money is considered as a desirable commodity and, therefore, more money is preferable to less money. Consequently, the utility function for money would always be upward sloping. However, as the utility function depicts the subjective attitude of a decision-maker to risk, the utility functions for different kinds of decision-makers would have different slopes. In this context, the decision-makers are classified in accordance with their psychological reactions to risk into three classes: risk-averse, risk-seekers, and risk-neutral.

A risk-averse is a person who is always willing to accept a small cash-certain amount than the expected values of a bet. Most of the people fall in this category. Our above analysis of derivation of utility function has been for a risk-averse individual. The utility function for such a person would be, like in Figure 13.3, the one shown in Figure 13.4.

For a risk-seeker, a person who demands an amount of cash-certain in excess of the expected monetary pay-off, the utility function would be rising at an increasing rate. Thus, a risk taker would gain more satisfaction from a bet where he can have a Rs 50,000 win (which has 0.5 chance of occurrence) or a loss of Rs 10,000 with an equal probability, than a cash-certain amount of Rs 20,000—the expected pay-off of the bet. He may demand, say, Rs 25,000 in cash to leave the bet.

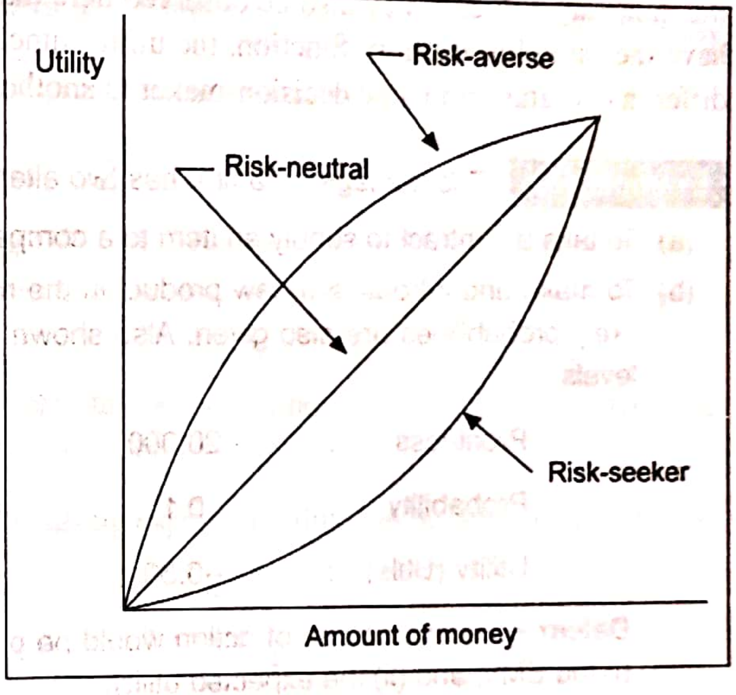


Figure 13.4 *Utility Functions*

Also shown in the figure in reference is the utility function of a risk-neutral, the one who is neutral between a cash-certain amount and a bet whose expected value is equal to that. For instance, if a person is neutral between a sure payment of a sum of Rs 20,000 and a bet with a gain of Rs 50,000 and a loss of Rs 10,000 with equal probabilities, he is said to be risk-neutral. A risk-neutral is neither averse to risk nor does he encourage it. For such decision-maker, the utility function is always linear.

The utility functions of risk-averse, risk-seeker and risk-neutral decision-makers are reproduced in Figure 13.5. It may be observed from part *a* of the figure that, if *M* be a certain sum of money the utility of an amount of double than that represented by *M*, equal to 2*M*, is less than twice the utility of the amount *M*. Thus, $U_{2M} < 2U_M$. Similarly, from the parts *b* and *c*, respectively, it is clear that for a risk-seeker $U_{2M} > 2U_M$ while $U_{2M} = 2U_M$ for a risk-neutral individual.

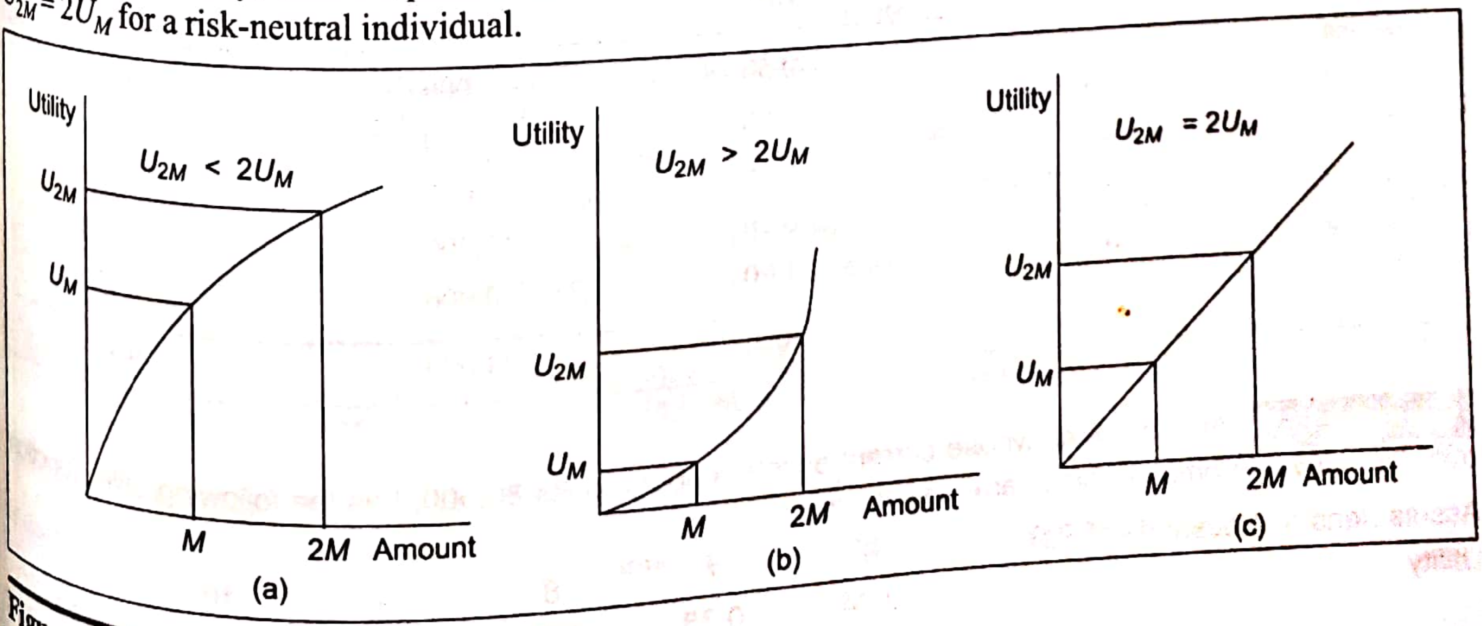


Figure 13.5 *Utility Functions*

The increasing slope of the utility function of a risk-seeker indicates that he has increasing marginal utility of money. Thus when his cash position improves, he places more value on each additional rupee that he would get. For a risk-averse, on the other hand, the marginal utility of money is decreasing. A linear utility function for a risk-neutral implies a constant marginal utility of money for him. His utility varies in direct proportion to the monetary value. It may also be observed here that while the risk-neutral decision makers would always have the same linear utility function, the utility functions for the risk-evaders or risk-preferers are likely to differ in curvature from one decision-maker to another.

Example 13.5 The manager of a firm has two alternatives to choose from, for the next quarter.

(a) To take a contract to supply an item to a company which would result in a sure profit of Rs 20,000.

(b) To make and introduce a new product in the market. The likely profit/loss possibilities along with the likely probabilities are also given. Also shown are the utility values associated with the various profit levels.

Profit/loss	:	–20,000	0	20,000	40,000	80,000
Probability	:	0.1	0.2	0.3	0.3	0.1
Utility (Utils)	:	–0.50	0	0.45	0.70	1.20

Determine which course of action would be preferred by the manager when he wanted to maximise (i) the EMV, and (ii) the expected utility.

(i) The expected profit associated with the alternative (a) is given to be Rs 20,000 while for the alternative (b) it equals Rs 24,000, as given in Table 13.17. Thus, the manager should decide for introducing the new product if he is seeking to maximise expected monetary value.

(ii) The expected utility, *EU*, for alternative (b) is 0.415 as shown in the table, while the utility associated with the profit of Rs 20,000 associated with alternative (a) is 0.45. Therefore, if the decision-maker seeks to maximise expected utility, he would decide in favour of the alternative (a).

TABLE 13.17 Calculation of EMV and EU

Profit/loss (i)	*Prob. (ii)*	*Utility (iii)*	*Expected Profit (i) × (ii)*	*Expected Utility (ii) × (iii)*
–20,000	0.1	–0.50	–2,000	–0.050
0	0.2	0	0	0.000
20,000	0.3	0.45	6,000	0.135
40,000	0.3	0.70	12,000	0.210
80,000	0.1	1.20	8,000	0.120
		Total	24,000	0.415

Example 13.6 An individual, whose current assets amount to Rs 80,000, has the following utility function over the relevant portion of his overall assets scale.

Assets (tens of thousands of Rs)	:	6	7	8	9	10	11
Utility	:	0.24	0.38	0.50	0.60	0.67	0.72

(a) He is offered a bet in which he has a sixty per cent chance of gaining Rs 20,000 and a forty per cent chance of losing Rs 20,000. Should he accept the offer?

(b) Alternatively, he is offered participation in two bets each involving a gain of Rs 10,000 with a probability of 0.6 and a loss of Rs 10,000 with a forty per cent chance. Should he decide differently than in case (a)?

(a) For a bet involving a gain of Rs 20,000 and a loss of Rs 20,000, we can calculate the expected utility as follows:

Outcome	*Assets*	*Utility*	*Probability*	*Expected Utility*
Gain	100,000	0.67	0.6	0.402
Loss	60,000	0.24	0.4	0.096
			Total	0.498

Presently, he has a utility of 0.50, while if he accepts the bet, his expected utility is 0.498. Thus, he would do well not to accept the bet.

(b) When he is invited to participate in two bets, the resulting expected utility can be obtained as follows:

Outcome	*Assets*	*Utility*	*Probability*	*Expected Utility*
Loses both	60,000	0.24	0.16 ($= 0.4 \times 0.4$)	0.0384
Loses one	80,000	0.50	0.48 ($= 2 \times 0.6 \times 0.4$)	0.2400
Wins both	100,000	0.67	0.36 ($= 0.6 \times 0.6$)	0.2412
			Total	0.5196

Since the expected utility is in excess of 0.50 (the utility corresponding to his present assets level), he should accept the offer.

REVIEW ILLUSTRATIONS

Example 13.7 The relevant data for three alternatives to invest Rs one lakh are given below:

Investment		*Relevant Data*
A (Dealership)	:	The probability of success in exporting a product is 0.7 and in that case the return could be Rs 40,000 per year. Otherwise it could be sold in domestic market with a profit of Rs 20,000 per year. In any case, the expenditure per year is Rs 10,000.
B (Property)	:	If an expenditure of Rs 25,000 is incurred for renovation, a net profit of Rs 50,000 could be made. For this, the probability is 0.8. Otherwise it can be sold as it is, for a profit of Rs 30,000. The whole transaction would take one year from the date of investment.
C (Fixed deposit)	:	The company in which investment is made would pay 25% if it runs successfully for a year. Otherwise it would pay the principal amount only. The probability for failure of the company is 0.2.

Which investment should be chosen and what would be the rate of return? *(ICWA, December, 1993)*

For each of the investments, the expected percentage return is shown calculated below:

Investment	Return	Probability	Expected Value	% Return
A	Rs 40,000	0.7	28,000	
	Rs 20,000	0.3	6,000	
			34,000	
			– 10,000	
			24,000	24%
B	Rs 25,000	0.8	20,000	
	Rs 30,000	0.2	6,000	
			26,000	26%
C	Rs 25,000	0.8	20,000	
	Rs 0	0.2	0	
			20,000	20%

Notes:

1. For investment *A*, expected value of return is Rs 34,000. With a given expenditure of Rs 10,000, the net return works out to be Rs 24,000 or 24%.
2. The return of Rs 25,000 on investment *B* is calculated as the profit on sale *less* cost of innovation = Rs 50,000 – Rs 25,000.
3. A 25% return on Rs 100,000 in investment *C* would mean Rs 25,000. In case of company failure, no return is indicated. Thus, expected return is $0.8 \times 25{,}000 + 0.2 \times 0 =$ Rs 20,000.

Conclusion: It is evident from the above calculations that investment *B* is the best one.

Example 13.8 An investor is given the following investment alternatives and percentage rates of return.

	States of Nature (Market Conditions)		
	Low	Medium	High
Regular shares	2%	5%	8%
Risky shares	–5%	7%	15%
Property	–10%	10%	20%

Over the past 300 days, 150 days have been medium market conditions and 60 days have had high market increases.

On the basis of these data, state the optimal investment strategy for the investor.

According to the given information, the probabilities of low, medium, and high market conditions would be 90/300 or 0.30, 150/300 or 0.50, and 60/300 or 0.20, respectively. The expected pay-offs for each of the alternatives are calculated and shown in Table 13.18.

TABLE 13.18 Determination of Expected Return

Market Conditions	Prob.	Strategy		
		Regular Shares	Risky Shares	Property
Low	0.30	0.02	−0.05	−0.10
Medium	0.50	0.05	0.07	0.10
High	0.20	0.08	0.15	0.20
Expected Return		0.053	0.050	0.060

Since the expected return of 6 per cent is the highest for property, the investor should invest in this alternative.

Example 13.9 Informatics Corporation summarises international information reports (on a weekly basis), prints sophisticated data and forecasts, which are purchased weekly by mutual funds, banks and insurance companies. This information is very expensive and the demand for the reports is limited to a maximum of 30 units. The possible demands are 0, 10, 20 and 30 reports per week. The profit per report sold is Rs 30 and the loss per report unsold is Rs 20. No production of extra reports during a week is possible. Further, there is a penalty cost of Rs 250 for not meeting the demand. Unsold reports cannot be carried on to the next week. Using the pay-off table, find out the number of reports to be produced if:

(i) Maximin or pessimistic strategy is adopted.

(ii) Maximax or optimistic strategy is used. *(CA, May, 1991)*

From the given information,

Profit per report sold = Rs 30

Loss per report unsold = Rs 20

Penalty for not meeting the demand = Rs 250

Using these values, the pay-off matrix can be derived as shown in Table 13.19.

TABLE 13.19 Pay-off Matrix

Demand: No. of Reports per week	Number of Reports Produced			
	0	10	20	30
0	0	−200	−400	−600
10	−250	300	100	−100
20	−250	50	600	400
30	−250	50	350	900
Minimum Pay-off	−250	−200	−400	−600
Maximum Profit	0	300	600	900

(i) The maximum of the minimum values is –200. Hence, according to the *maximin* strategy, the decision is to produce 10 copies of the report with a loss of Rs 200.

(ii) The maximum among the maximum values is 900 when 30 reports are produced per week. Thus, according to the *maximax* strategy, the decision is to produce 30 reports.

Example 13.10 ABC Company needs to increase its production beyond its existing capacity. It has narrowed the alternatives to two approaches to increase the production capacity:

(1) expansion at a cost of Rs 8 million, or

(2) modernisation, at a cost of Rs 5 million.

Both approaches would require the same amount of time for implementation. Management believes that over the required payback period, demand will either be high or moderate. Since high demand is considered to be somewhat less likely than moderate demand, the probability of high demand has been set at 0.35. If the demand is high, expansion would gross an additional Rs 12 million but modernisation only an additional amount of Rs 6 million, due to lower maximum production capacity. On the other hand, if the demand is moderate, the comparable figures would be Rs 7 million for expansion and Rs 5 million for modernisation.

(i) Calculate the conditional profit in relation to various action-and-outcome combination and states of nature.

(ii) If the company wants to maximise its EMV, should it modernise or expand?

(iii) Calculate the EVPI and EOL. *(MBA, Delhi, April 1...)*

From the given information, the conditional profit matrix is obtained as shown in Table 13.20.

TABLE 13.20 Conditional Profit (in millions of Rs)

Demand	Probability	Course of Action	
		Expand	*Modernise*
High	0.35	4	1
Moderate	0.65	(1)	0
Expected Profit		0.75	0.35

From the expected values, it is evident that the company should decide to expand.

Calculation of EVPI

$$\text{EVPI} = \text{EPPI} - \text{Expected profit for optimal decision}$$

$$\text{EPPI} = 0.35 \times 4 + 0.65 \times 0 = \text{Rs } 1.4 \text{ (million)}$$

$$\therefore \quad \text{EVPI} = \text{Rs } 1.4 - \text{Rs } 0.75 = \text{Rs } 0.65 \text{ (million)}$$

Calculation of EOL

We can obtain opportunity loss by subtracting every value in a row from the larger value in the row. For strategy of expansion, we get 0 and 1; while for the other strategy, we obtain 3 and 0. Accordingly,

EOL for Expansion : $0 \times 0.35 + 1 \times 0.65 = 0.65$

EOL for Modernisation : $3 \times 0.35 + 0 \times 0.65 = 1.05$

Example 13.11 A stockist of a particular commodity makes a profit of Rs 30 on each sale made within the same week of purchase, otherwise he incurs a loss of Rs 30 on each item. The data on the past sales are given here:

No. of items sold within the same week	:	5	6	7	8	9	10	11
Frequency	:	0	9	12	24	9	6	0

(a) Find out the optimum number of items the stockist should buy every week in order to maximise the profit.

(b) Calculate the expected value of perfect information. *(CA, May, 1996)*

(a) Since the number of items sold in a week are between 6 and 10, we may consider stocking 6 through 10 units as the courses of action. Also, the given frequencies may be converted into probabilities by dividing each of them by their sum total, that is 60. Now, using the given data, the conditional pay-off matrix is given in Table 13.21.

TABLE 13.21 Conditional Pay-off Matrix

No. of Units Sold	*Freq.*	*Prob.*	*No. of Units Stocked*				
			6	*7*	*8*	*9*	*10*
6	9	0.15	180	150	120	90	60
7	12	0.20	180	210	180	150	120
8	24	0.40	180	210	240	210	180
9	9	0.15	180	210	240	270	240
10	6	0.10	180	210	240	270	300
Expected Value			180	201	210	195	171

The expected pay-off for each of the actions is shown in the last row of the table. It is highest for the strategy of stocking 8 units. Thus, optimal strategy: stock 8 units.

(b) To obtain EVPI, we first calculate expected pay-off under perfect information (EPPI) as follows:

$$\text{EPPI} = 0.15 \times 180 + 0.20 + 210 + 0.40 \times 240 + 0.15 \times 270 + 0.10 \times 300 = \text{Rs } 235.50$$

$$\therefore \quad \text{EVPI} = \text{EPPI} - \text{Expected pay-off under optimal policy} = \text{Rs } 235.50 - \text{Rs } 210 = \text{Rs } 25.50$$

Example 13.12 The Rs 8,00,000 property of the Goodwill India Co has one-tenth of one per cent chance of catching fire that will cause damage to the property to the extent of Rs 1,00,000; and a one-twentieth of one per cent chance of catching fire that will completely destroy the property. The management of the company decides to insure the property and is reviewing two alternative insurance policies:

(a) A policy with Rs 50,000-deductible, that is, the insurance company covers all losses expect the initial Rs 50,000. The annual premium for such a policy is known to be one-tenth of one per cent of the value of the property.

(b) A no-deduction policy with full compensation having an annual premium of Rs 1,000.

If the company's objective is cost minimisation, which policy should it opt for? Sketch both, the pay-off table and the opportunity loss table, for the situation and solve both of them.

According to the given information, the management has two choices open:

A_1 : The Rs 50,000-deductible policy, and

A_2 : The full compensation policy

The various states of nature and their probabilities are:

State of Nature, E_i	Prob., p_i
E_1: A damage of Rs 1,00,000	0.0010
E_2: A damage of Rs 8,00,000	0.0005
E_3: No damage	0.9985

The pay-offs associated with different act-event combinations are contained in Table 13.22 and the expected pay-offs are calculated.

TABLE 13.22 Calculation of Expected Pay-offs

Event	Probability	Act, A_j	
		A_1	A_2
E_1	0.0010	49,200	99,000
E_2	0.0005	749,200	799,000
E_3	0.9985	(800)	(1,000)
Expected Pay-off		(375)	(500)

From the table we observe that the expected pay-offs associated with the acts are: A_1 (375) and A_2 (500). The negative pay-offs here imply the cost. Since the cost for A_1 is lower, the management should go in for the Rs 50,000-deductible policy. The opportunity loss values, derived from the pay-off values, are given in Table 13.23 and the expected values are also calculated.

TABLE 13.23 Calculation of Expected Opportunity Loss

Event E_i	Probability	Act, A_j	
		A_1	A_2
E_1	0.0010	49,800	0
E_2	0.0005	49,800	0
E_3	0.9985	0	200
EOL		74.7	199.7

A_1 is the optimal act (as before), since the expected opportunity loss, EOL, is lower for it.

Example 13.13 Leisure Hotels Ltd. is planning to build another 700-room complex. It has been suggested that because the existing hotels average only 70% occupancy on an annual basis, the new complex should have only 500 rooms. It has been estimated that the cost per room per annum is Rs 21,000.

following data based on demand at similar complexes has been obtained:

	Number of Days	Daily Demand	Average Price per Occupied Room per Day
Peak season	200	800	Rs 100
In between	80	600	Rs 80
Slack season	85	500	Rs 60

ow,

(a) prepare a pay-off table for a complex with 500, 600, 700 and 800 rooms;

(b) on the basis of this data, advise management as to the number of rooms it should include;

(c) state with reasons two further categories of data that would be useful before making the final decision.

(ICMA, May 1983, Adapted)

(a) For each of the different number of rooms, say R, the annual cost would be given by 21,000 R while the annual revenue could be broken down according to the particular season. Thus, if $R = 500$, all rooms would be let in each season and all revenues will be:

Peak season $200 \times 500 \times 100 =$ Rs 100,00,000

In between $80 \times 500 \times 80 =$ Rs 32,00,000

Slack season $85 \times 500 \times 60 =$ Rs 25,50,000

The revenues in each season can be calculated for each of different sizes of the complex to get the pay-off table (Table 13.24).

TABLE 13.24 Pay-off Table (Rupees in lac)

Season	No. of Rooms			
	500	600	700	800
Peak	100.0	120.0	140.0	160.0
In between	32.0	38.4	38.4	38.4
Slack	25.5	25.5	25.5	25.5

(b) The annual profits for each room size are as follows:

$R = 500$: Profit $= 100.0 + 32.0 + 25.5 - 105.0 =$ Rs 52.5 lac

$R = 600$: Profit $= 120.0 + 38.4 + 25.5 - 126.0 =$ Rs 57.9 lac

$R = 700$: Profit $= 140.0 + 38.4 + 25.5 - 147.0 =$ Rs 56.9 lac

$R = 800$: Profit $= 160.0 + 38.4 + 25.5 - 168.0 =$ Rs 55.9 lac

The size of complex yielding the maximum expected annual profit is 600 rooms with an associated value of Rs 57,90,000 per annum.

(c) The data on demand, cost and revenue are seemingly based on current estimates only. In keeping with the nature of investment, it would be advisable to estimate the likely trends in these figures to ascertain the profitability in the years to come.

Besides, the revenues are presumably based on only the occupied rooms, having no regard to the unoccupied rooms. It may be possible, for instance, to stimulate extra demand by providing convention facilities and thereby attracting revenue from rooms which otherwise would be empty.

Example 13.14 The annual demand for a seasonal product follows the distribution shown here.

Demand (units) :	3,000	3,500	4,000	4,500	5,000
Probability :	0.10	0.20	0.30	0.30	0.10

The manufacturer of this item can produce it by one of the three methods:

(a) Using the existing equipment at a cost of Rs 8 per unit.
(b) Buy special equipment for Rs 22,000 whose salvage value at the end of the year would be Rs 2,000. The variable cost per unit, using this equipment is Rs 2.
(c) Buy special equipment for Rs 90,000, which would be depreciated on straight-line basis over a period of 4 years. The variable cost using this equipment is Rs 1.20 per unit.

Which method of production should the manufacturer follow in order to maximise profit, assuming that production must meet all the demand?

Since the cost data are given in this problem, we shall determine the conditional cost values and then calculate the expected cost in respect of each of the methods of production. The method with the least expected cost would be chosen. The calculations are given in Table 13.25.

TABLE 13.25 Calculation of Expected Cost

Event	*Prob.*	*Courses of Action, A_j*		
		A_1 : Existing	*A_2 : Special Equip. I*	*A_3 : Special Equip. II*
E_1 : 3,000	0.10	24,000	26,000	26,100
E_2 : 3,500	0.20	28,000	27,000	26,700
E_3 : 4,000	0.30	32,000	28,000	27,300
E_4 : 4,500	0.30	36,000	29,000	27,900
E_5 : 5,000	0.10	40,000	30,000	28,500
Expected cost		32,400	28,100	27,360

Result: Buy special equipment with a 4-year life.

Example 13.15 A company is considering the introduction of a new product to its existing product range. It has defined two levels of sales as 'high' and 'low' on which to base its decision and has estimated the chances that each market level will occur, together with their costs and consequent profits or losses. This information is summarised below:

States of Nature	*Probability*	*Course of Action*	
		Market Product (Rs '000)	*Do Not Market Product (Rs '000)*
High sales	0.3	150	0
Low sales	0.7	−40	0

The company's marketing manager suggests a market research survey be undertaken to provide further information on which to base the decision. On past experience with a certain market research organisation, the marketing manager assesses its ability to give good information in the light of subsequent actual sales achievements as follows:

Market Research Survey Outcome	Actual Sales	
	Market 'High'	Market 'Low'
'High' sales forecast	0.5	0.1
'Indecisive' survey report	0.3	0.4
'Low' sales forecast	0.2	0.5

Given that to undertake the market research survey will cost Rs 20,000, determine whether or not there is a case for employing the market research organisation. *(MBA, Delhi, October, 1996)*

If the survey is not undertaken, the expected profit is:

$$0.3 \times 150 + 0.70 \times (-40) = 17 \text{ that is, Rs } 17{,}000$$

If the survey is undertaken, the following outcomes are possible, with probabilities as shown:

Actual sales	*Survey outcome*	*Probability*
High	High	$0.3 \times 0.5 = 0.15$
High	Indecisive	$0.3 \times 0.3 = 0.09$
High	Low	$0.3 \times 0.2 = 0.06$
Low	High	$0.7 \times 0.1 = 0.07$
Low	Indecisive	$0.7 \times 0.4 = 0.28$
Low	Low	$0.7 \times 0.5 = 0.35$

For the three possible survey outcomes,

$$P(\text{High}) = 0.15 + 0.07 = 0.22$$

$$P(\text{Indecisive}) = 0.09 + 0.28 = 0.37$$

$$P(\text{Low}) = 0.06 + 0.35 = 0.41$$

If the survey were to give a *high sales forecast*, then the probabilities of sales actually being high and low are:

$$P(\text{High Sales/High Forecast}) = 0.15/0.22 = 0.682$$

$$P(\text{Low Sales/High Forecast}) = 0.07/0.22 = 0.318$$

and the expected profit is $0.682 \times 150 + 0.318 \times (-40) = 89.58$

For an indecisive forecast:

$$P(\text{High Sales/Indecisive Forecast}) = 0.09/0.37 = 0.243$$

$$P(\text{Low Sales/Indecisive Forecast}) = 0.28/0.37 = 0.757$$

and the expected profit is $0.243 \times 150 + 0.757 \times (-40) = 6.17$

For a low forecast:

P(High Sales/Low Forecast) = 0.06/0.41 = 0.146

P(Low Sales/Low Forecast) = 0.35/0.41 = 0.854

and the expected profit is $0.146 \times 150 + 0.854 \times (-40) = -12.26$

Since this last value is negative, it implies that the company would expect to make a loss if it launched the new product following a low sales forecast. It should not, therefore, go ahead and, consequently, in this case the expected profit would be taken to be zero. Now, taking all possible survey forecasts into account, the overall expected profit is

$$0.22 \times 89.58 + 0.37 \times 6.17 + 0.41 \times 0 = 22.0 \text{ or Rs } 22{,}000.$$

Therefore, as a result of taking the survey, expected profit has increased from Rs 17,000 to Rs 22,000. As the survey costs Rs 20,000 to undertake, it is evidently not worthwhile.

Example 13.16 An ice cream manufacturer sells soft scoop ice cream in special pressurised containers and is planning production for the summer, which is the peak period. The company wishes to ensure that it has the best quantity of containers on hand: too few and sales will be lost; too many and the surplus will have to be stored over the winter at a substantial cost. The containers can only be purchased in lots of 500. The following table shows the estimated lost contributions for various ordering patterns:

	Number of New Containers Bought			
	0	*500*	*1,000*	*1,500*
Poor summers—low sales	0	20	20	30
Fair summer—reasonable sales	15	0	15	20
Good summer—good sales	20	20	0	15
Very good summer—very high sales	30	25	15	0

Based on past data, the probabilities of the different types of weather are:

Poor : 0.3 Fair : 0.4 Good : 0.2 Very good : 0.1

The firm has obtained a copy of the long-range weather forecast for the summer which indicates that there will be good summer, but past experience states that the forecasts are not 100% accurate, as follows:

P(forecast *good* but weather *poor*) = 0.3

P(forecast *good* but weather *fair*) = 0.4

P(forecast *good* and weather *good*) = 0.7

P(forecast *good* but weather *very good*) = 0.2

You are required

(a) to calculate the number of containers that should be purchased based on past data only (i.e. ignore the forecast);

(b) to calculate whether the decision in (a) would need to be altered if the forecast is taken into account;

(c) to explain any changes made in the purchase decision as a result of comparing your answers to (a) and (b) above.

(ICMA, November, 1993, adapted)

(a) *When the decision is on the basis of past data only: prior analysis*

The conditional loss contributions are given in Table 13.26. Also, the expected loss contributions are shown calculated. From the calculations, it is evident that 500 new pressurised containers should be ordered.

TABLE 13.26 Calculation of Expected Loss Contribution (Prior)

States of Nature	*Probability*	*Course of Action: No of Containers* 0	500	1,000	1,500
Poor Summer	0.3	0	20	20	30
Fair Summer	0.4	15	0	15	20
Good Summer	0.2	20	20	0	15
Very Good Summer	0.1	30	25	15	0
Expected Value		13	12.5	13.5	20

(b) *When decision is based on forecasts: posterior analysis*

We may first calculate the probability of poor, fair, good and very good summer, given that the forecast is good (E), as $P(A/E)$, $P(B/E)$, $P(C/E)$ and $P(D/E)$, respectively, using Bayes' theorem as follows: Given $P(A) = 0.3$, $P(E/A) = 0.3$, $P(B) = 0.4$, $P(E/B) = 0.4$, $P(C) = 0.2$, $P(E/C) = 0.7$, $P(D) = 0.1$, and $P(E/D) = 0.2$. Accordingly,

$$P(E) = P(A \cap E) + P(B \cap E) + P(C \cap E) + P(D \cap E)$$
$$= P(A) \times P(E/A) + P(B) \times (E/B) + P(C) \times (E/C) + P(D) \times (E/D)$$
$$= 0.3 \times 0.3 + 0.4 \times 0.4 + 0.2 \times 0.7 + 0.1 \times 0.2 = 0.41$$

Now,

$$P(A/E) = P(A \cap E)/P(E) = 0.3 \times 0.3/0.41 = 0.22$$
$$P(B/E) = P(B \cap E)/P(E) = 0.4 \times 0.4/0.41 = 0.39$$
$$P(C/E) = P(C \cap E)/P(E) = 0.2 \times 0.7/0.41 = 0.34$$
$$P(D/E) = P(D \cap E)/P(E) = 0.1 \times 0.2/0.41 = 0.05$$

We may now recalculate the expected values using these posterior probabilities. This is shown in Table 13.27.

TABLE 13.27 Calculation of Expected Loss Contribution (Posterior)

State of Nature	*Probability*	*Course of Action: No. of Containers* 0	500	1,000	1,500
Poor Summer	0.22	0	20	20	30
Fair Summer	0.39	15	0	15	20
Good Summer	0.34	20	20	0	15
Very Good Summer	0.05	30	25	15	0
Expected Value		14.15	12.45	11.0	19.50

It is clear from the table that since the expected lost contribution is the lowest with 1,000, the order should be in the lots of 1,000.

(c) Evidently, the purchasing decision is changed now in view of the increased ability of forecast good weather. It has resulted in deciding to purchase 1,000 pressurised containers in the expectation of a good summer associated with good sales.

Example 13.17 An engineering company's plant maintenance budget for the month of May is showing an adverse cost variance of Rs 1,200 on a total budget of Rs 20,000. The factory manager is uncertain whether it would be worthwhile investigating the cost variance and taking any corrective action since he estimates this will cost Rs 1,000. Further, he estimates that given the usual variation on his maintenance costs, he would only have a 50% chance in any case of being in the range of Rs 19,500 – Rs 20,500.

Higher than planned maintenance costs could be due to engineering failures more complex than expected. These would be rectified after investigation. The manager estimates that consequential losses of not investigating such a situation could amount to Rs 4,000.

(a) Calculate the standard deviation of the estimated maintenance costs;

(b) set out the pay-off table for action on the decision as to investigation of variance;

(c) state the decision rule and the probability that the process is out of control which would lead to a decision to investigate;

(d) advise the manager as to whether he should investigate the causes of the variances in May budget. (Assume that the variances in the maintenance budget follow a normal probability pattern).

(ICMA, May, 1982, Adapted)

(a) Given that variances in the maintenance budget follow normal distribution, the maintenance cost would be normally distributed with mean, μ = Rs 20,000, as shown in Figure 13.6. Further, since there is a 50 per cent chance that the cost would be between Rs 19,500 and Rs 20,500, there is a 0.25 probability that its value would lie between Rs 20,000 and Rs 20,500.

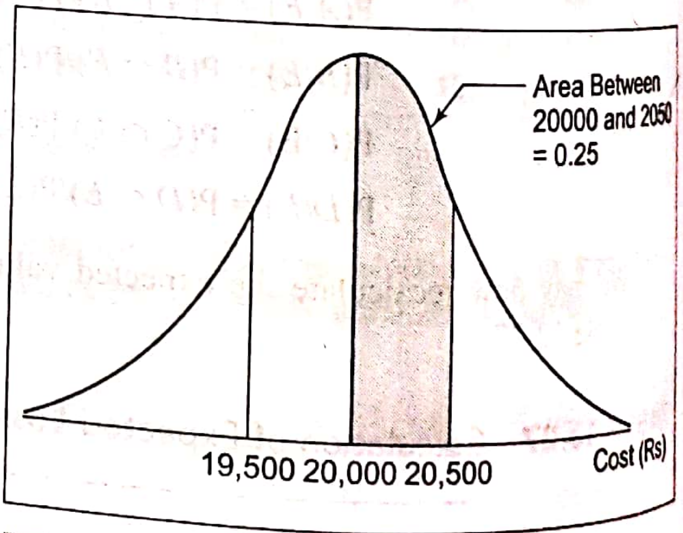


Figure 13.6 *Distribution of Cost*

We know,

$$z = \frac{X - \mu}{\sigma}$$

From the normal real table, the z-value corresponding to area = 0.25 is 0.675. Thus, we have

$$0.675 = \frac{20{,}500 - 20{,}000}{\sigma}$$

or

$$\sigma = \frac{500}{0.675} = \text{Rs } 740$$

(b) There are two possible states of nature, or events:

E_1—the process is out of control (i.e. maintenance cost ≠ 20,000)

E_2—the process is in control (i.e. maintenance costs = 20,000)

Similarly, there are two courses of action, namely

A_1—investigate the variance,

A_2—do not investigate the variance.

According to the given information, the pay-off matrix is given here, the pay-offs being the cost values

TABLE 13.28 Pay-off Table

Event	Act, A_j	
	A_1	A_2
E_1	1,000	4,000
E_2	1,000	0

(c) If the decision rule to be used is that of minimising the expected cost, the optimal course of action would depend upon the likelihood of each event. Let p be the probability of E_1, so that

$$P(E_1) = p \quad \text{and} \quad P(E_2) = 1 - p.$$

The expected values for each of the acts would be as follows:

$$E(A_1) = 1{,}000p + 1{,}000(1-p) = 1{,}000$$

$$E(A_2) = 4{,}000p + 0(1-p) = 4{,}000p.$$

For indifference,

$$1{,}000 = 4{,}000p \quad \text{or} \quad p = 1{,}000/4{,}000 = 0.25.$$

Thus, if the probability that the process is out of control exceeds 0.25, then the variance calls for investigation.

(d) When the process is under control, the variances in the cost will follow normal distribution (as given) with mean = 0 and a standard deviation = Rs 740. The probability of observing a variance of upto Rs 1,200 can be determined as follows:

$$z = \frac{1{,}200 - 0}{740} = 1.62$$

Area beyond $z = 1.62$ equals 0.053. Thus, the probability that a cost variance as great as Rs 1,200 (favourable or adverse) will occur = $2 \times 0.053 = 0.11$ or 11%. It suggests that the variance should be investigated as the low probability of 11% is indicative of the state that the process might be out of control or at least that the probability of its being out of control is greater than 0.25.

Note here the probability value 0.11 should not be taken to mean that it represents the conditional probability of observing a variance in excess of Rs 1,200 *assuming* that the process is in control. If we denote the event process in control by C and 'process not in control' by $\bar{C}$, we have,

$$P(\text{var.} = 1{,}200/C) = 0.11.$$

To obtain the probability that the process is in control having observed the variance of Rs 1,200, we need to use the Bayes' Theorem as follows:

$$P(C/\text{var.} = 1{,}200) = \frac{P(\text{var.} = 1{,}200/C) \times P(C)}{P(\text{var.} = 1{,}200/C) \times P(C) + P(\text{var.} = 1{,}200/\bar{C}) \times P(\bar{C})}$$

However, it is possible to calculate this probability only if we knew $P(C)$ and $P(\text{var.} = 1{,}200/\bar{C})$.

Example 13.18 The investment staff of TNC Bank is considering four investment proposals for a client shares, bonds, real estate and savings certificate. These investments will be held for one year. The past data regarding the four proposals are given below:

Shares: There is 25 per cent chance that shares will decline by 10 per cent, a 30 per cent chance that they will remain stable and a 45 per cent chance that they will increase in value by 15 per cent. Also, the shares under consideration do not pay any dividends.

Bonds: These bonds stand a 40 per cent chance of increase in value by 5 per cent and 60 per cent chance of remaining stable and they yield 12 per cent.

Real Estate: This proposal has a 20 per cent chance of increasing 30 per cent in value, a 25 per cent chance of increasing in 20 per cent value, a 40 per cent chance of increasing in 10 per cent value, a 10 per cent chance of remaining stable and a 5 per cent chance of losing 5 per cent of its value.

Savings Certificates: These certificates yield 8.5 per cent with certainty.

Use a decision tree to structure the alternatives available to the investment staff, and using the expected value criterion, choose the alternative with the highest expected value.

(MBA, Delhi, Nov. 2006)

The tree diagram is shown in Figure 13.7.

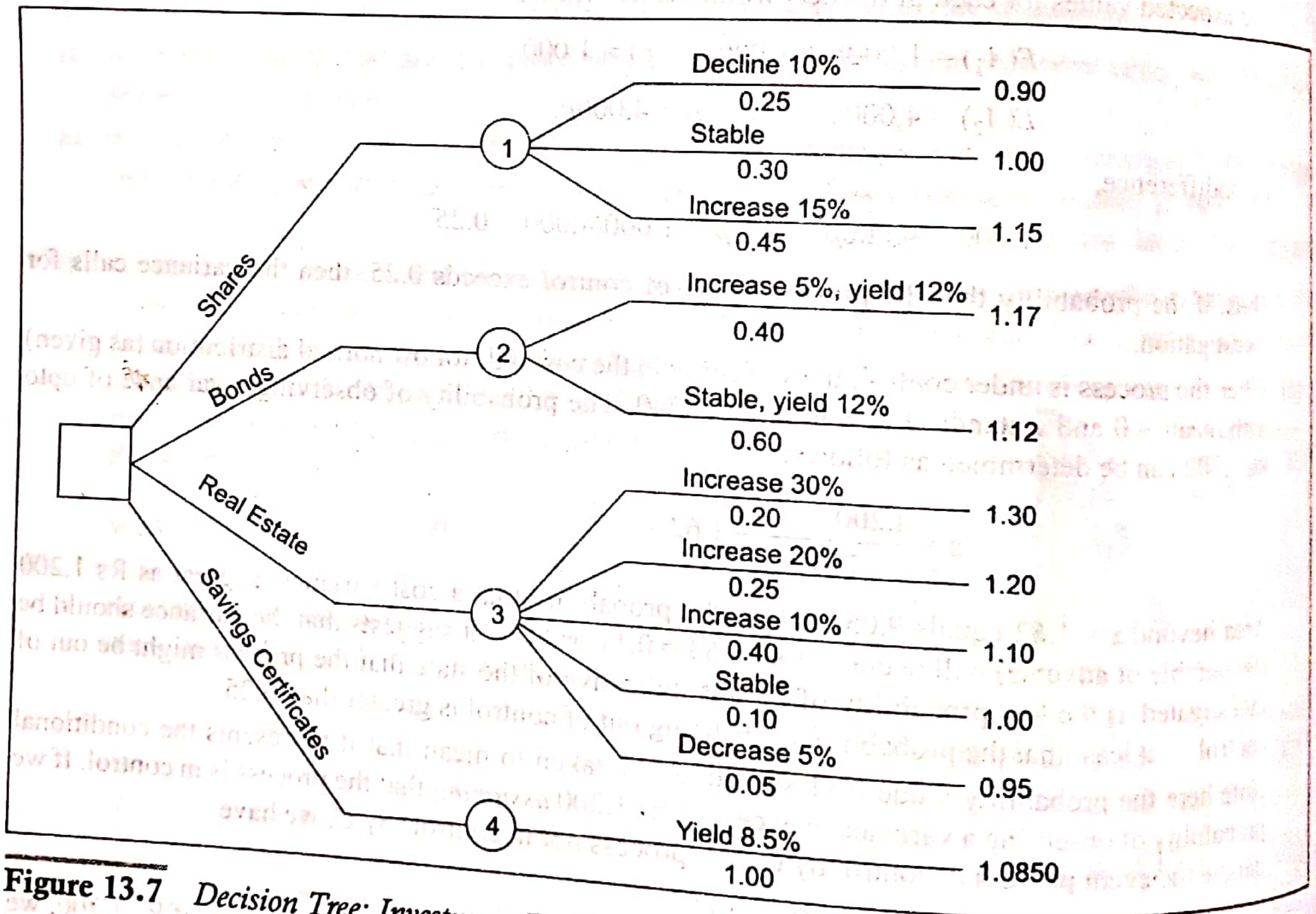


Figure 13.7 *Decision Tree: Investment Problem*

The end values give the amount a rupee 1 invested would become in each case.

The expected value at each chance node is shown calculated below:

Node 1 : $0.25 \times 0.90 + 0.30 \times 1.00 + 0.45 \times 1.15 = 1.0425$

Node 2 : $0.40 \times 1.17 + 0.6 \times 1.12 = 1.1400$

Node 3 : $0.20 \times 1.30 + 0.25 \times 1.20 + 0.40 \times 1.10 + 0.10 \times 1.00 + 0.05 \times 0.95 = 1.1475$

Node 4 : $1.00 \times 1.0850 = 1.0850$

It is evident that the maximum expected pay-off is at node 3. Hence, investment should be made in real estate.

Example 13.19 A Finance Manager is considering drilling a well. In the past, only 70% of wells drilled were successful at 20 metres depth in that area. Moreover, on finding no water at 20 metres, some persons in that area drilled it further up to 25 metres but only 20% struck water at that level. The prevailing cost of drilling is Rs 500 per metre. The Finance Manager estimated that in case he does not get water in his own well, he will have to pay Rs 15,000 to buy water from outside for the same period of getting water from the well. The following decisions are considered:

(i) Do not drill any well;

(ii) Drill up to 20 metres, and

(iii) If no water is found at 20 metres, drill further upto 25 metres.

Draw an appropriate decision tree and determine the Finance Manager's optimal strategy.

(CA, May, 1992)

Based on the given information, the decision tree is shown in Figure 13.8.

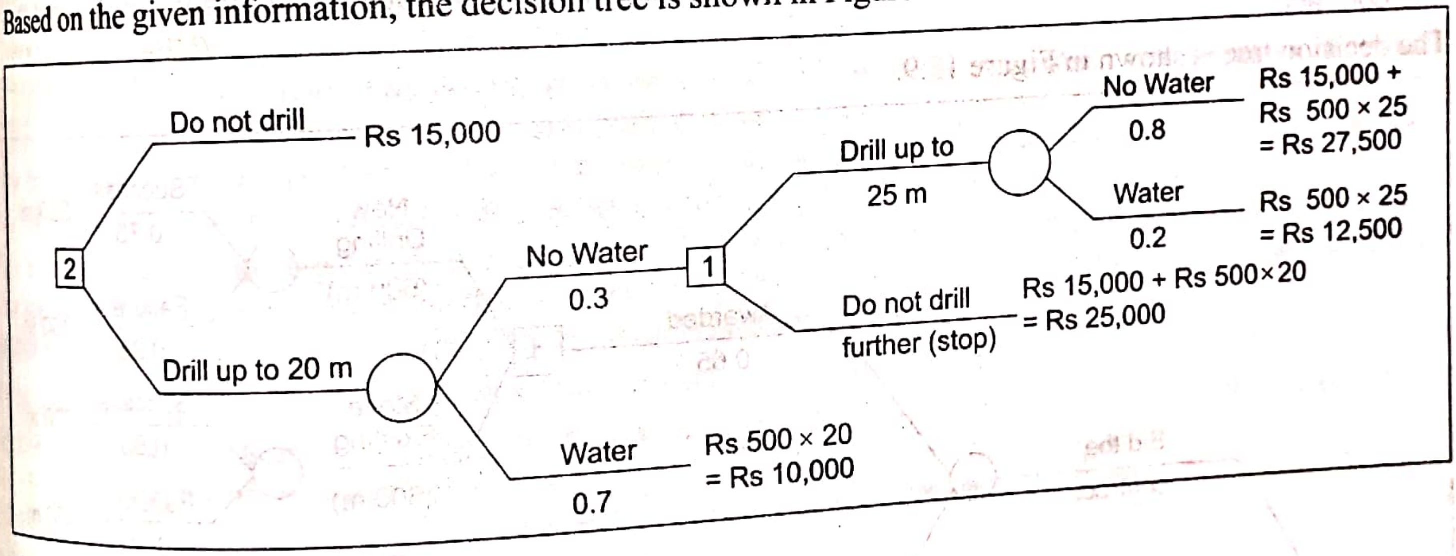


Figure 13.8 *Decision Tree: Drilling Problem*

The analysis of the tree is given in Table 13.29.

TABLE 13.29 Analysis Table: Decision Tree

Decision Node	*Options*	*Expected Cost*	*Decision*
1	Drill up to 25 metres	0.8 × 27,500 + 0.2 × 12,500 = Rs 24,500	Drill up to 25 metres
	Stop	Rs 25,000	
2	Do not drill	Rs 15,000	Drill up to 20 metres
	Drill up to 20 metres	0.3 × 24,500 + 0.7 × 10,000 = Rs 14,350	

From the analysis table, it may be observed that decision at node 2 implies that if it is decided to drill up to 20 metres and water is not found, then drilling up to 25 metres should be done. At node 1, the decision taken is to drill up to 20 metres as it involved lower expected cost. Thus, the optimal strategy is to drill up to 20 metres and if water is not struck then drill further to 25 metres.

Example 13.20 The Oil India Corporation is considering whether to go for an offshore drilling contract to be awarded in Bombay High. If they bid, value would be Rs 600 million with 65% chance of gaining the contract. The Corporation may set up a new drilling operation or move already existing operation, which has proved successful, to new site. The probability of success and expected returns are as follows:

Outcome	New Drilling Operation: Probability	New Drilling Operation: Expected Revenue (Rs million)	Existing Operation: Probability	Existing Operation: Expected Revenue (Rs million)
Success	0.75	800	0.85	700
Failure	0.25	200	0.15	350

If the Corporation does not bid or lose the contract, they can use Rs 600 million to modernise their operations. This would result in a return of either 5% or 8% on the sum invested with probabilities 0.45 and 0.55, respectively.

(a) Construct a decision tree for the problem showing clearly the courses of action.

(b) By applying an appropriate decision criterion recommend whether or not the Corporation should bid the contract.

(MBA, Delhi, 2005)

The decision tree is shown in Figure 13.9.

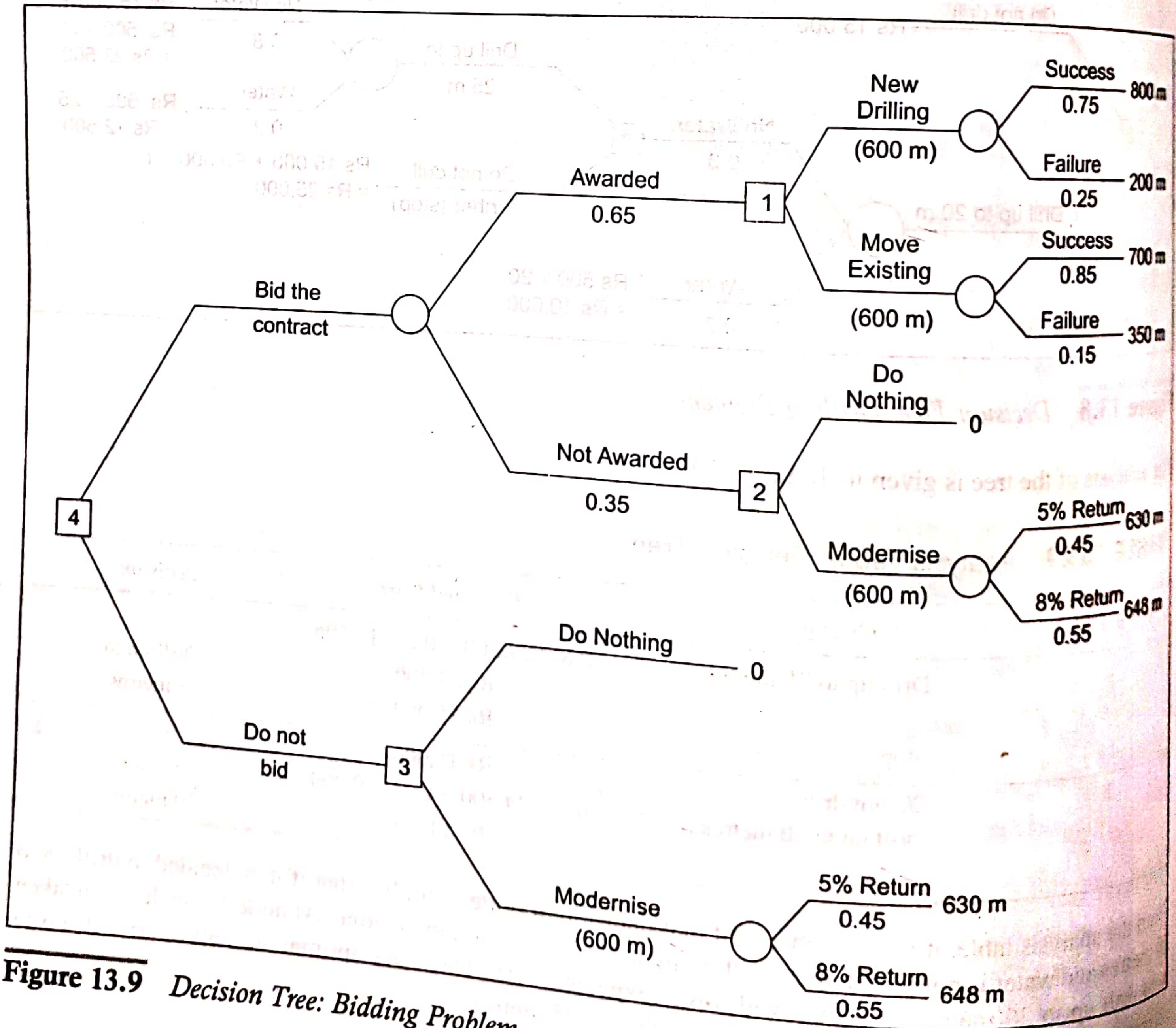


Figure 13.9 *Decision Tree: Bidding Problem*

The decision tree is analysed below:

Decision Node	Options	EMV	Decision
1	(a) New drilling	$0.75 \times 800 + 0.25 \times 200 - 600 = 50$ m	New drilling
	(b) Move existing operations	$0.85 \times 700 + 0.15 \times 350 - 600 = 47.5$ m	
2	(a) Do nothing	0	Modernise
	(b) Modernise	$0.45 \times 630 + 0.55 \times 648 - 600 = 39.9$ m	
3	(a) Do nothing	0	Modernise
	(b) Modernise	$0.45 \times 630 + 0.55 \times 648 - 600 = 39.9$ m	
4	(a) Bid the contract	$0.65 \times 50 + 0.35 \times 39.9 = 46.465$ m	Bid the contract
	(b) Do not bid the contract	39.9 m	

Decision: Bid for the contract. If the contract is awarded, then set up new drilling operation. If not, then modernise.

Example 13.21 A company has developed a new product in its R&D laboratory. The company has the option of setting up production facility to market this product straight away. If the product is successful, then over the three years expected product life, the returns will be Rs 120 lakh with a probability of 0.70. If the market does not respond favourable, then the returns will be only Rs 15 lakh with probability of 0.30.

The company is considering whether it should test market this product building a small pilot plant. The chance that the test market will yield favourable response is 0.80. If the test market gives favourable response, then the chance of successful total market improves to 0.85.

If the test market gives poor response then the chance of success in the total market is only 0.30. As before, the returns from a successful market will be Rs 120 lakh and from an unsuccessful market only Rs 15 lakh. The installation cost to produce for the total market is Rs 40 lakh and the cost of the test marketing pilot plant is Rs 5 lakh. Using decision-tree analysis, draw a decision-tree diagram, carry out necessary analysis to determine the optimal decisions. *(MBA, Delhi, Nov. 2008)*

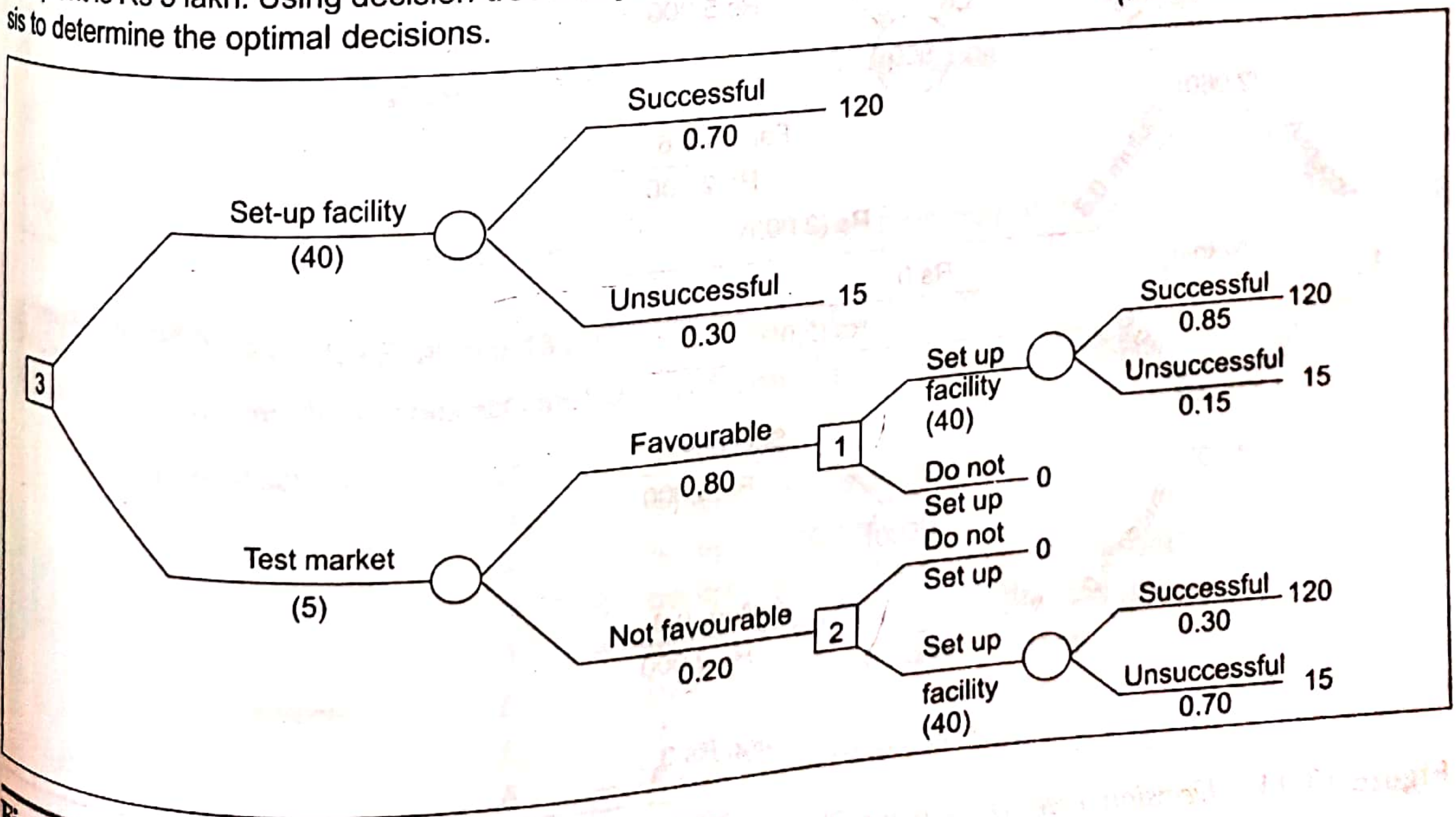


Figure 13.10 *Decision Tree: Setting up Facility*

Using the given information, the decision tree is shown in Figure 13.10. Note that, to begin with, the company has two options—to set up production facility or go for test market. In case it decides for test market, the company may set up the facility or quit, whether the result of test marketing is favourable or not. The decision tree is analysed below.

Decision Node	Options	EMV (in lakh of Rs)	Decision
1	Set up facility Do not set up	0.85 × 120 + 0.15 × 15 – 40 = 64.25 0	Set up facility
2	Set up facility Do not set up	0.30 × 120 + 0.70 × 15 – 40 = 6.50 0	Set up facility
3	Set up facility Test market	0.70 × 120 + 0.30 × 15 – 40 = 48.5 0.80 × 64.25 + 0.2 × 6.50 – 5 = 47.7	Set up facility

Thus, the company should set up production facility straight way and not undertake test market.

Example 13.22 A businessman has two independent investments *A* and *B* available to him, but he lacks the capital to undertake both of them simultaneously. He can choose to take *A* first and then stop, or if *A* is successful then take *B*, or vice versa. The probability of success on *A* is 0.7, while for *B* it is 0.4. Both investments require an initial capital outlay of Rs 2,000, and both return nothing if the venture is unsuccessful. Successful completion of *A* will return Rs 3,000 (over cost), successful completion of *B* will return Rs 5,000 (over cost). Draw the decision tree and determine the best strategy. *(CA, May, 1985)*

The decision tree corresponding to the given information is depicted in Figure 13.11.

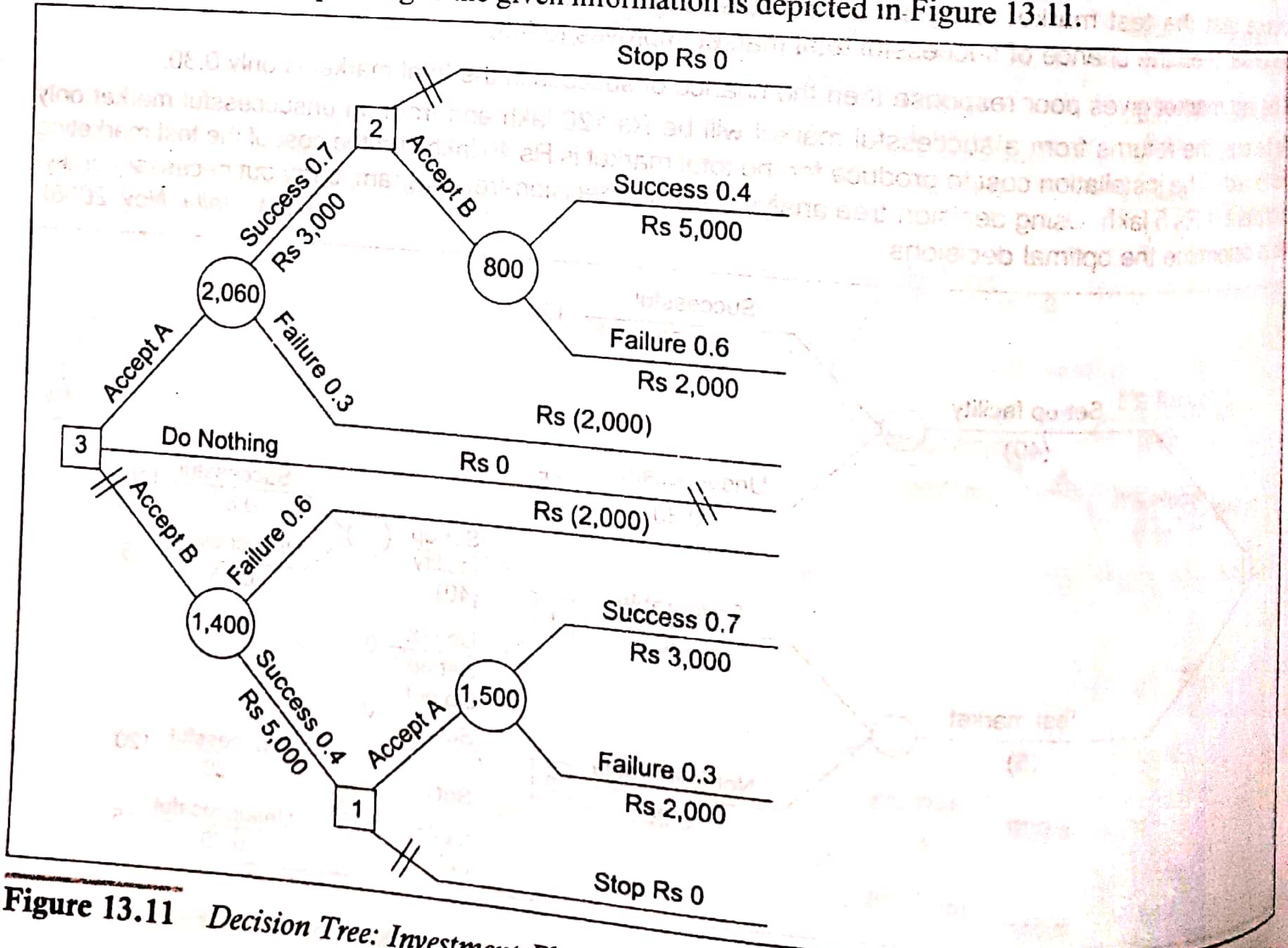


Figure 13.11 *Decision Tree: Investment Plan*

According to this, there are five strategies: (a) do nothing, (b) accept A and then stop, (c) accept B and then stop, (d) accept A and, if successful, then accept B, and (e) accept B and, if successful, accept A. The expected pay-off values are indicated in the circles representing chance nodes. There are three decision points marked 1, 2 and 3.

The evaluation of decision points by calculating expected values is contained in Table 13.30. The optimal decision is to accept A and, if successful, then accept B.

TABLE 13.30 Evaluation of Decision Points

Decision Point	*Outcome*	*Probability*	*Conditional Value*	*Expected Value*
1. Accept A	Success	0.7	Rs 3,000	2,100
	Failure	0.3	Rs (2,000)	(600)
				1,500
Stop				0
2. Accept B	Success	0.4	Rs 5,000	2,000
	Failure	0.6	Rs (2,000)	(1,200)
				800
Stop				0
3. Accept A	Success	0.7	Rs 3,000 + 800	2,660
	Failure	0.3	Rs (2,000)	(600)
				2,060
Accept B	Success	0.4	Rs 5,000 + 1,500	2,600
	Failure	0.6	Rs (2,000)	(1,200)
				1,400
Do nothing				0

Example 13.23 Re-solve Example 13.22 by preparing a pay-off table.

For the given problem, the various acts and states of nature, the events, are given here:

Courses of action:
- A_1 : do nothing
- A_2 : accept A and then stop
- A_3 : accept B and then stop
- A_4 : accept A and, if successful, then accept B
- A_5 : accept B and, if successful, then accept A

Events:
- E_1 : both A and B are successful
- E_2 : A will be successful but not B
- E_3 : B will be successful but not A
- E_4 : neither A nor B will be successful

The probabilities of various events are:

$E_1 : 0.7 \times 0.4 = 0.28$; $E_2 : 0.7 \times 0.6 = 0.42$; $E_3 : 0.3 \times 0.4 = 0.12$, and $E_4 : 0.3 \times 0.6 = 0.18$.

The conditional pay-offs, resulting from different combinations of actions and events are given in Table 13.31. Since the expected value for the act A_4 is the largest, it represents the optimal choice.

TABLE 13.31 Calculation of Expected Pay-offs

Event, E_i	Prob.	Act, A_j				
		A_1	A_2	A_3	A_4	A_5
E_1	0.28	0	3,000	5,000	8,000	8,000
E_2	0.42	0	3,000	(2,000)	1,000	(2,000)
E_3	0.12	0	(2,000)	5,000	(2,000)	3,000
E_4	0.18	0	(2,000)	(2,000)	(2,000)	(2,000)
Expected Pay-off		0	1,500	800	2,060	1,400

Example 13.24 Mr X is considering whether to make an investment in a project with the following likely returns:

Amount (Rs)	*Probability*
2,00,000	0.6
–40,000	0.4

The utility function of Mr X is approximated as follows:

$$U = -0.0003M^2 \quad \text{when} \quad M < -5{,}000$$
$$= 1.05M \quad \text{when} \quad M \geq -5{,}000$$

Should the project be undertaken by him? Consider (a) Expected monetary value criterion, and (b) Expected utility criterion.

Calculation of expected monetary value (EMV) and expected utility (EU) is shown in Table 13.32.

TABLE 13.32 Calculation of EMV and EU

Conditional Monetary Value (i)	*Conditional Utility* (ii)*	*Probability (iii)*	*Expected Monetary Value (i) × (iii)*	*Expected Utility (ii) × (iii)*
200,000	210,000	0.6	120,000	126,000
– 40,000	– 480,000	0.4	–16,000	–192,000
Total			104,000	–66,000

* Obtained by substituting monetary values in the utility function.

Since EMV is positive, the project should be undertaken according to the EMV criterion, while it should not be accepted on the basis of the EU criterion since EU is negative.

Theory of Games

INTRODUCTION

In business and economics literature, the term 'game' refers to the general situation of conflict and competition in which two or more competitors (or participants) are involved in decision-making activities in anticipation of certain outcomes over a period of time. The competitors are referred as *players.* A player may be an *individual,* a *group of individuals,* or an *organization.* Few examples of competitive and conflicting decision environment involving the interaction between two or more competitors where techniques of theory of games may be used to resolve them are: (i) pricing of products, where a firm's ultimate sales are determined not only by the price levels it selects but also by the prices its competitors set, (ii) various TV networks have found that program success is largely dependent on what the competitors presents in the same time slot; the outcomes of one networks programming decisions have, therefore, been increasingly influenced by the corresponding decisions made by other networks, (iii) success of a business tax strategy depends greatly on the position taken by the internal revenue service regarding the expenses that may be disallowed, (iv) success of an advertising/marketing campaign depends largely on various types of services offered to the customers, etc.

The models in the *theory of games* can be classified depending upon the following factors:

Number of players: If a game involves only two players (competitors), then it is called a *two-person game.* However, if the number of players is more, the game is referred to as *n-person game.*

Sum of gains and losses: If in a game sum of the gains to one player is exactly equal to the sum of losses to another player, so that sum of the gains and losses equals zero, then the game is said to be a *zero-sum game.* Otherwise it is said to be *non-zero sum game.*

Strategy: The strategy for a player is the list of all possible actions (moves or courses of action) that he will take for every pay-off (outcome) that might arise. It is assumed that the rules governing the choices are known in advance to the players. The outcome resulting from a particular choice is also known to the players in advance and is expressed in terms of numerical values (e.g. money, per cent of market share or utility). Here it is not necessary that players have definite information about each other's strategies.

The particular strategy (or complete plan) by which a player optimizes his gains or losses without knowing the competitor's strategies is called *optimal strategy.* The expected outcome when players follow their optimal strategy is called the *value of the game* and is generally denoted by V.

Generally, two types of strategies are employed by players in a game.

(a) *Pure Strategy:* It is the decision rule which is always used by the player to select the particular strategy (course of action). Thus, each player knows in advance of all strategies out of which he always selects only one particular strategy regardless of the other player's strategy, and the objective of the players is to maximize gains or minimize losses.

(b) *Mixed Strategy:* Courses of action that are to be selected on a particular occasion with some fixed probability are called *mixed strategies.* Thus, there is a probabilistic situation and objective of the players is to maximize expected gains or to minimize expected losses by making choice among pure strategies with fixed probabilities.

Mathematically, a mixed strategy for a player with two or more possible courses of action is the set S of n non-negative real numbers (probabilities) whose sum is unity, n being the number of pure strategies of the player. If p_j $(j = 1, 2, \ldots, n)$ is the probability with which the pure strategy, j would be selected, then,

$$S = \{p_1, p_2, \ldots, p_n\}\ ;\ p_1 + p_2 + \ldots + p_n = 1\ ;\ p_j \geq 0 \text{ of all } j.$$

Remark: If a particular $p_j = 1$ $(j = 1, 2, \ldots, n)$ and all others are zero, the player is said to select pure strategy j. A flow chart of using game theory approach to solve a problem is shown in Fig. 1.

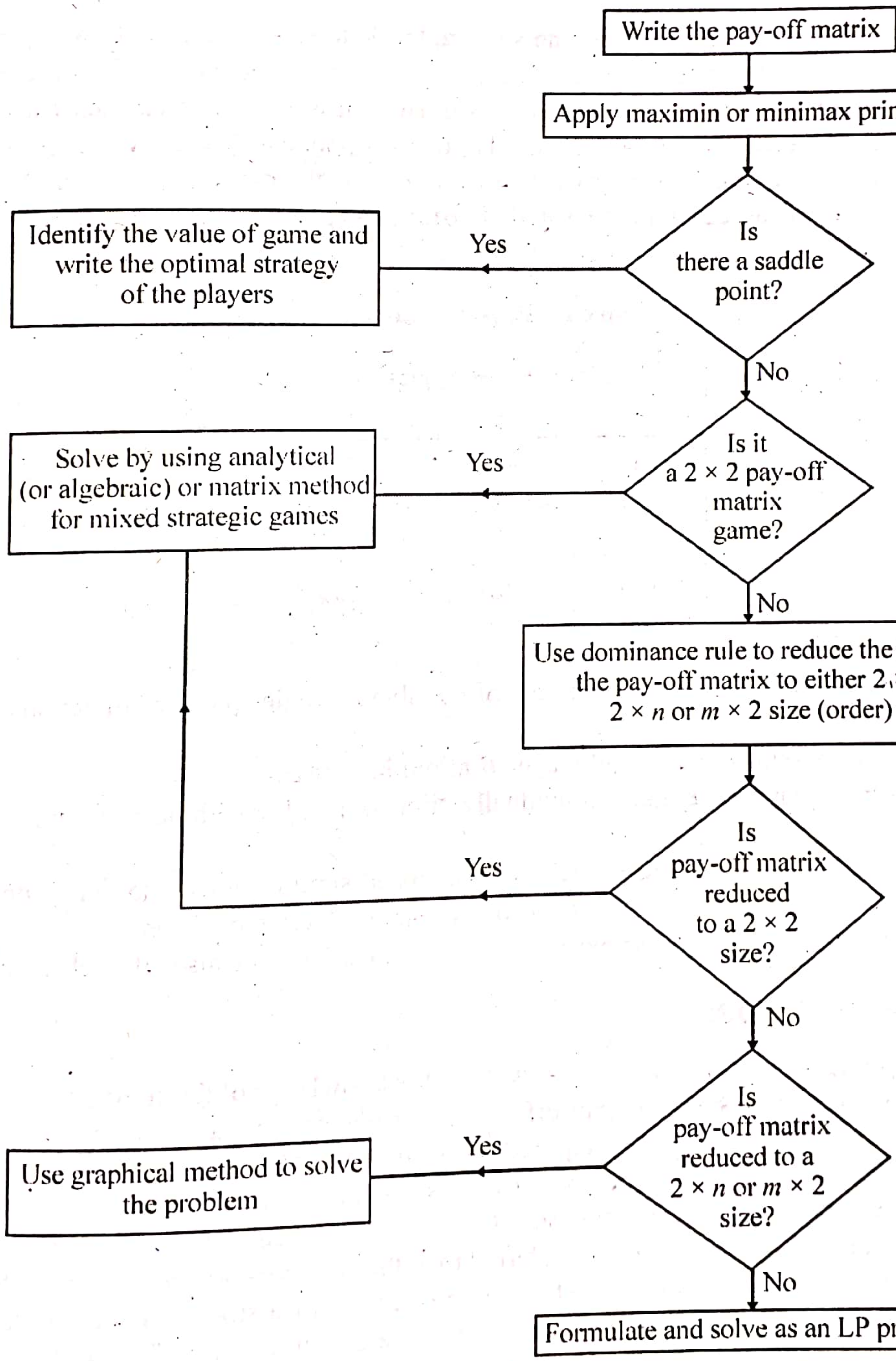


Fig. 1 Flow Chart of Game Theory Approach

TWO-PERSON ZERO-SUM GAMES

Pay-off matrix: The pay-offs (a quantitative measure of satisfaction a player gets at the end of the play) in terms of gains or losses, when players select their particular strategies (courses of action), can be represented in the form of a matrix, called the pay-off matrix. Since the game is zero-sum, the gain of one player is equal to the loss of other and vice versa. In other words, one player's pay-off table would contain the same amounts in pay-off table of other player with the sign changed. Thus, it is sufficient to construct pay-off table only for one of the players.

If player A has m strategies represented by the subscripted letters: $A_1, A_2, \ldots, A_m$ and player B has n strategies represented by the subscripted letters: $B_1, B_2, \ldots, B_n$. The numbers m and n need not be equal. The total number of possible outcomes is therefore $m \times n$. Here, it is assumed that each player knows not only his own list of possible courses of action but also of his opponent. For convenience, it is assumed that player A is always a gainer whereas player B a loser. Let a_{ij} be the pay-off which player A gains from player B if player A chooses strategy i and player B chooses strategy j. Then the pay-off matrix is shown in the Table 1.

Table 1 Pay-off Matrix

Player A's Strategies	Player B's Strategies B_1	B_2	$\ldots$	B_n
A_1	a_{11}	a_{12}	$\ldots$	a_{1n}
A_2	a_{21}	a_{22}	$\ldots$	a_{2n}
$\vdots$	$\vdots$	$\vdots$		$\vdots$
A_m	a_{m1}	a_{m2}	$\ldots$	a_{mn}

Assumptions of the Game

1. Each player has available to him a finite number of possible strategies (courses of action). The list may not be the same for each player.
2. Player A attempts to maximize gains and player B minimize losses.
3. The decisions of both players are made individually prior to the play with no communication between them.
4. The decisions are made simultaneously and also announced simultaneously so that neither player has an advantage resulting from direct knowledge of the other player's decision.
5. Both the players know not only possible pay-offs to themselves but also of each other.

GAMES WITH SADDLE POINT

The selection of an optimal strategy by each player without the knowledge of the competitor's strategy is the basic problem of playing games. Since the pay-offs for either player provide all the essential information, therefore, only one player's pay-off table is required to evaluate the decisions. By convention, the pay-off table for the player whose strategies are represented by rows (say player A) is constructed. Now the objective of the study is to know how these players must select their respective strategies so that they may optimize their pay-off. Such a decision-making criterion is referred to as the *minimax-maximin principle.* Such principle in pure strategies game always leads to the best possible selection of a strategy for both players.

If the maximin value = minimax value, then the game is said to have a *saddle (equilibrium) point* and the corresponding strategies are called *optimal strategies.* The amount of pay-off, i.e. V at an equilibrium point is known as the *value of the game.* A game may have more than one saddle point.

Rules to Determine Saddle Point

1. Select the minimum (lowest) element in each row of the pay-off matrix and write them under 'row minima' heading. Then select the largest element among these elements and enclose it in a rectangle, $\square$.
2. Select the maximum (largest) element in each column of the pay-off matrix and write them under 'column maxima' heading. Then select the lowest element among these elements and enclose it in a circle, $\bigcirc$.
3. Find out the element(s) which is same in the circle as well as rectangle and mark the position of such element(s) in the matrix. This element represents the value of the game and is called the saddle (or equilibrium) point.

GAME WITHOUT SADDLE POINT

In certain cases, there is no pure strategy solution for a game, i.e. no saddle point exists. In all such cases, to solve games both the players must determine an optimal mixture of strategies to find a saddle (equilibrium) point.

The optimal strategy mixture for each player may be determined by assigning to each strategy its probability of being chosen. The strategies so determined are called *mixed strategies* because they are probabilistic combination of available choices of strategy.

Remark: For solving a 2 × 2 game without saddle point, the following formula is also used. If pay-off matrix for player A is given by

$$\text{Player } A \overset{\text{Player } B}{\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}}$$

then following formulae are used to find the value of game and optimal strategies.

$$p_1 = \frac{a_{22} - a_{21}}{a_{11} + a_{22} - (a_{12} + a_{21})}; \qquad q_1 = \frac{a_{22} - a_{12}}{a_{11} + a_{22} - (a_{12} + a_{21})}$$

where $$p_2 = 1 - p_1; \quad q_2 = 1 - q_1 \quad \text{and} \quad V = \frac{a_{11}a_{22} - a_{21}a_{12}}{a_{11} + a_{22} - (a_{12} + a_{21})}$$

Algebraic Method This method is used to determine probability of using different strategies by players A and B. This method becomes quite lengthy when number of strategies for both the players are more than two. Consider a game where pay-off matrix is: $[a_{ij}]_{m \times n}$. Let $(p_1, p_2, \ldots, p_m)$ and $(q_1, q_2, \ldots, q_n)$ be the probabilities with which players A and B select their strategies $(A_1, A_2, \ldots, A_m)$ and $(B_1, B_2, \ldots, B_n)$, respectively. If V is the value of game, then expected gain to player A when player B selects strategies $B_1, B_2, \ldots, B_n$ one by one is given by left hand side of the following simultaneous equations, respectively. Since player A is the gainer player and expects at least V, therefore, we must have

Player B

Player A	B_1	B_2	$\ldots$	B_n	*Probability*
A_1	a_{11}	a_{12}	$\ldots$	a_{1n}	p_1
A_2	a_{21}	a_{22}	$\ldots$	a_{2n}	p_2
$\vdots$	$\vdots$				$\vdots$
A_m	a_{m1}	a_{m2}	$\ldots$	a_{mn}	p_m
Probability	q_1	q_2	$\ldots$	q_n	

$$a_{11} p_1 + a_{21} p_2 + \ldots + a_{m1} p_m \geq V$$
$$a_{12} p_1 + a_{22} p_2 + \ldots + a_{m2} p_m \geq V$$
$$\vdots \qquad \vdots \qquad \vdots \qquad\qquad (1)$$
$$a_{1n} p_1 + a_{2n} p_2 + \ldots + a_{mn} p_m \geq V$$

where $p_1 + p_2 + \ldots + p_m = 1$ and $p_i \geq 0$ for all i

Similarly, the expected loss to player B when player A selects strategies $A_1, A_2, \ldots, A_m$ one by one can also be determined. Since player B is the loser player, therefore, he must have,

$$a_{11} q_1 + a_{12} q_2 + \ldots + a_{1n} q_n \leq V$$
$$a_{21} q_1 + a_{22} q_2 + \ldots + a_{2n} q_n \leq V$$
$$\vdots \qquad \vdots \qquad \vdots \qquad\qquad (2)$$
$$a_{m1} q_1 + a_{m2} q_2 + \ldots + a_{mn} q_n \leq V$$

where $q_1 + q_2 + \ldots + q_n = 1$ and $q_j \geq 0$ for all j.

To get the values of p_i's and q_j's, above inequalities are considered as equations and are then solved for given unknowns. However, if the system of equations so obtained is inconsistent, then at least one of the inequalities must hold as strict inequality. The solultion can now be obtained only by applying trial and error method.

Graphical Method The graphical method is useful for the game where the pay-off matrix is of the size $2 \times n$ or $m \times 2$, i.e. the game with mixed strategies that has only two undominated pure strategies for one of the players in the two-person zero-sum game.

Optimal strategies for both the players assign no-zero probabilities to the same number of pure strategies. Therefore, if one player has only two strategies, the other will also use the same number of strategies. Hence, this method is useful in finding out which of the two strategies can be used.

Consider the following $2 \times n$ pay-off matrix of a game without saddle point.

	Player B				
Player A	B_1	B_2	$\ldots$	B_n	*Probability*
A_1	a_{11}	a_{12}	$\ldots$	a_{1n}	p_1
A_2	a_{21}	a_{22}	$\ldots$	a_{2n}	p_2
Probability	q_1	q_2	$\ldots$	q_n	

Player A has two strategies A_1 and A_2 with probability of their selection p_1 and p_2, respectively, such that $p_1 + p_2 = 1$ and $p_1, p_2 \geq 0$. Now for each of the pure strategies available to player B, expected pay off for player A would be as follows:

B's Pure Strategies	*A's Expected Pay-off*
B_1	$a_{11} p_1 + a_{21} p_2$
B_2	$a_{12} p_1 + a_{22} p_2$
$\vdots$	$\vdots$
B_n	$a_{1n} p_1 + a_{2n} p_2$

According to the maximin criterion for mixed strategy games, player A should select the value of probability p_1 and p_2 so as to maximize his minimum expected pay-offs. This may be done by plotting the straight lines representing player A's expected pay-off values.

The highest point on the lower boundary of these lines will give maximum expected pay-off among the minimum expected payoffs and the optimum value of probability p_1 and p_2.

Now the two strategies of player B corresponding to those lines which pass through the maximin point can be determined. It helps in reducing the size of the game to (2×2), which can be easily solved by any of the methods discussed earlier.

The $(m \times 2)$ games are also treated in the same way except that the upper boundary of the straight lines corresponding to B's expected pay-off will give the maximum expected pay-off to player B and the lowest point on this boundary will then give the minimum expected pay-off (minimax value) and the optimum value of probability q_1 and q_2.

Linear Programming Method The two-person zero-sum games can also be solved by linear programming. The major advantage of using linear programming technique is to solve mixed-strategy games of larger dimension pay-off matrix.

To illustrate the transformation of a game problem to a linear programming problem, consider a pay-off matrix of size $m \times n$. Let a_{ij} be the element in the ith row and jth column of game pay-off matrix, and letting p_i be the probabilities of m strategies ($i = 1, 2, \ldots, m$) for player A. Then the expected gains for player A, for each of player B's strategies will be

$$V = \sum_{i=1}^{m} p_i \, a_{ij}\,, \quad j = 1, 2, \ldots, n$$

The aim of player A is to select a set of strategies with probability p_i ($i = 1, 2, \ldots, m$) on any play of game such that he can maximize his minimum expected gains.

Now to obtain values of probability p_i, the value of the game to player A for all strategies by player B must be at least equal to V. Thus to maximize the minimum expected gains, it is necessary that

$$\begin{aligned} a_{11} p_1 + a_{21} p_2 + \ldots + a_{m1} p_m &\geq V \\ a_{12} p_1 + a_{22} p_2 + \ldots + a_{m2} p_m &\geq V \\ \vdots \qquad\qquad & \quad \vdots \\ a_{1n} p_1 + a_{2n} p_2 + \ldots + a_{mn} p_m &\geq V \end{aligned}$$

where $\quad p_1 + p_2 + \ldots + p_m = 1;\ p_i \geq 0 \quad$ for all i.

Dividing both sides of the m inequalities and equation by V the division is valid as long as $V > 0$. In case $V < 0$, the direction of inequality constraints must be reversed. But if $V = 0$, division would be meaningless. In this case a constant can be added to all entries of the matrix ensuring that the value of the game (V) for the revised matrix becomes more than zero. After optimal solution is obtained, the true value of the game is obtained by substracting the same constant value. Let $p_i/V = x_i$, (≥ 0). Then we have

$$\begin{aligned} a_{11} \frac{p_1}{V} + a_{21} \frac{p_2}{V} + \ldots + a_{m1} \frac{p_m}{V} &\geq 1 \\ a_{12} \frac{p_1}{V} + a_{22} \frac{p_2}{V} + \ldots + a_{m2} \frac{p_m}{V} &\geq 1 \\ \vdots \qquad\qquad & \quad \vdots \\ a_{1n} \frac{p_1}{V} + a_{2n} \frac{p_2}{V} + \ldots + a_{mn} \frac{p_m}{V} &\geq 1 \\ \frac{p_1}{V} + \frac{p_2}{V} + \ldots + \frac{p_m}{V} &= 1 \end{aligned}$$

Since the objective of player A is to maximize the value of the game, V which is equivalent to minimizing $1/V$. Similarly, player B is to minimize the expected loss V, which is equivalent to maximizing $1/V$. The resulting linear programming problems can be stated as

Player A

Minimize $Z_p\ (= 1/V) = x_1 + x_2 + \ldots + x_m$

subject to the constraints

$$a_{11}\, x_1 + a_{21}\, x_2 + \ldots + a_{m1}\, x_m \geq 1$$
$$a_{12}\, x_1 + a_{22}\, x_2 + \ldots + a_{m2}\, x_m \geq 1$$
$$\vdots$$
$$a_{1n}\, x_1 + a_{2n}\, x_2 + \ldots + a_{mn}\, x_m \geq 1$$

and $x_1, x_2, \ldots, x_m \geq 0$

where $x_i = p_i / V \geq 0\ ;\ i = 1, 2, \ldots, m$

Player B

Maximize $Z_q\ (= 1/V) = y_1 + y_2 + \ldots + y_n$

subject to the constraints

$$a_{11}\, y_1 + a_{12}\, y_2 + \ldots + a_{1n}\, y_n \leq 1$$
$$a_{21}\, y_1 + a_{22}\, y_2 + \ldots + a_{2n}\, y_n \leq 1$$
$$\vdots$$
$$a_{m1}\, y_1 + a_{m2}\, y_2 + \ldots + a_{mn}\, y_n \leq 1$$
$$y_1, y_2, \ldots, y_n \geq 0$$

where, $y_j = q_j / V \geq 0;\ \ j = 1, 2, \ldots, n$

It may be noted that the LP problem for player B is the dual of LP problem for player A and vice versa. Therefore, solution of the dual problem can be obtained from the primal simplex table. Since for both the players $Z_p = Z_q$, the expected gain to player A in the game will be exactly equal to expected loss to player B.

Remark: Linear programming technique requires all variables to be non-negative and therefore to obtain a non-negative of value V of the game, the data to the problem, i.e. a_{ij} in the pay-off table should all be non-negative. If there are some negative elements in the pay-off table, a constant to every element in the pay-off table must be added so as to make the smallest element zero; the solution to this new game will give an optimal mixed strategy for the original game. The value of the original game then equals the value of the new game minus the constant.

THE RULES (PRINCIPLES) OF DOMINANCE

The rules of dominance are especially used for the evaluation of two-person zero-sum games without saddle (equilibrium) point. Certain dominance principles are stated as follows:

1. For player B, if each element in column, say C_r is greater than or equal to the corresponding element in another column, say C_s in the pay-off matrix, then the column C_r is dominant by column C_s (i.e. *r*th strategy is dominated by *s*th strategy) and therefore column C_r can be deleted from the pay-off matrix.
2. For player A, if each element in a row, say R_r is less than or equal to the corresponding element in another row, say R_s in the pay-off matrix, then the row R_r is dominated by row R_s and therefore row R_r can be deleted from the pay-off matrix.
3. A strategy say, k can also be dominated if it is inferior (less attractive) to an average of two or more other pure strategies. In this case, if the domination is strict, then strategy k can be deleted. If strategy k dominates the convex linear combination of some other pure strategies, then one of the pure strategies involved in the combination may be deleted. The domination will be decided as rules 1 and 2 above.

Remarks: 1. Rules (principles) of dominance discussed are used when the pay-off matrix is a profit matrix for the player A and a loss matrix for player B. Otherwise the principle gets reversed.

2. The value of the game, in general, satisfies the equation, maximin value $\leq V \leq$ minimax value.
3. A game is said to be a *fair game* if the lower (maximin) and upper (minimax) values of the game are equal and both equals zero.
4. A game is said to be *strictly determinable* if the lower (maximin) and upper (minimax) values of the game are equal and both equal the value of the game.

SOLVED EXAMPLES

Example 1 For the game with pay-off matrix:

Player A	Player B B_1	B_2	B_3
A_1	−1	2	−2
A_2	6	4	−6

determine the best strategies for players A and B. Also determine the value of game. Is this game (i) fair? (ii) strictly determinable?

[*Gujarat Univ., MSc (Stat.), 1983*]

Solution In this example, gains to player A or losses to player B are represented by the positive quantities whereas losses to A and gains to B are represented by negative quantities. It is assumed that A wants to maximize his minimum gains from B. Since the payoffs given in the matrix are what A receives, therefore, he is concerned with the quantities which represent the row minimums. Now A can do no worse than receive one of these values, and best of them occurs when he chooses strategy A_1. This choice provides a pay-off of − 2 to A when B chooses strategy B_3. This refers to A's choice of A_1 as his maximum pay-off strategy because this row contains the maximum of A's minimum possible pay-offs from his competitor B.

Player A	Player B B_1	B_2	B_3	Row minimum	
A_1	−1	2	(−2)	−2	← *Maximin*
A_2	6	4	− 6	− 6	
Column maximum	6	4	(−2) ← *Minimax*		

Similarly, it is assumed that B wants to minimize his losses and wishes that his losses to A be as small as possible. There are column maximums that represent the greatest payments B might have to make to A. The smallest of these losses is − 2, which occurs when A chooses his course of action, A_1 and B chooses his course of action, B_3. This choice of B_3 by B is his minimax loss strategy because this column amount is the minimum of the maximum possible losses.

In Table 2, the quantity − 2 in the A_1 row and B_3 column is enclosed both in box and circle. That is, it is both the minimum of the column maxima and the maximum of the row minima. This value is referred to as *saddle point*. The value of the game is, $V = -2$, for player A. The value of game is always expressed from the point of view of the player whose strategies are listed in the rows.

The game is strictly determinable. Also since the value of the game is not zero, the game is not fair.

Example 2 A company management and the labour union are negotiating a new three year settlement. Each of these has 4 strategies:

I : Hard and aggressive bargaining
II : Reasoning and logical approach
III : Legalistic strategy
IV : Conciliatory approach

The costs to the company are given for every pair of strategy choice.

Union Strategies	Company Strategies *I*	*II*	*III*	*IV*
I	20	15	12	35
II	25	14	8	10
III	40	2	10	5
IV	− 5	4	11	0

What strategy will the two sides adopt? Also determine the value of the game.

Solution Applying the rule of finding out the saddle point, we obtain the saddle point which is enclosed both in a circle and a rectangle as shown below:

Company Strategies

Union Strategies	*I*	*II*	*III*	*IV*	Row minimum
I	20	15	(12)	35	[12] ← *Maximin*
II	25	14	8	10	8
III	40	2	10	5	2
IV	– 5	4	11	0	– 5
Column maximum	40	15	(12)	35	
			↑ *Minimax*		

Since Maximin = Minimax = Value of game = 12, therefore the company will always adopt strategy III–Legalistic strategy and union will always adopt strategy I–Hard and aggressive bargaining.

Example 3 Find the range of values of p and q which will render the entry (2, 2) a saddle point for the game:

Player B

Player A	B_1	B_2	B_3
A_1	2	4	5
A_2	10	7	q
A_3	4	p	6

Solution First ignoring the values of p and q in the pay-off matrix, determine the maximin and minimax values in the usual manner as shown below:

Player B

Player A	B_1	B_2	B_3	Row minimum
A_1	2	4	5	2
A_2	10	[7]	q	[7] ← *Maximin*
A_3	4	p	(6)	4
Column maximum	10	7	(6) ← *Minimax*	

As shown above, since there exists no unique saddle point, therefore, saddle point will exist at the position (2, 2) only when $p \leq 7$ and $q > 7$.

Example 4 For what value of λ, the game with following pay-off matrix is strictly determinable?

Player B

Player A	B_1	B_2	B_3
A_1	λ	6	2
A_2	– 1	λ	– 7
A_3	– 2	4	λ

[*Bharthiar Univ., MSc (Maths) 1989*]

Solution First, ignoring the value of λ, determine the maximin and minimax values of the pay-off matrix, as shown below:

	Player *B*			
	B_1	B_2	B_3	Row minimum
A_1	λ	6	2	2 ← *Maximin*
A_2	–1	λ	–7	–7
A_3	–2	4	λ	–2
Column maximum	–1	6	2	
	↑ *Minimax*			

Since saddle point in the above table is not unique, the value of the game lies between –1 and 2, i.e. $-1 \le V \le 2$. For strictly determinable game, we must have $-1 \le \lambda \le 2$.

Example 5 Two players *A* and *B* match coins. If the coins match, then A wins two units of value, if the coins do not match, then *B* wins 2 units of value. Determine the optimum strategies for the players and the value of the game. [*Bombay Univ., BSc (App. Comp.) 1985*]

Solution The pay-off matrix for the matching player is,

	Player *B*	
Player *A*	*H*	*T*
H	2	–2
T	–2	2

The pay-off matrix has no saddle point. The optimum mixed strategies for players *A* and *B*, respectively are determined by

$$p_1 = \frac{a_{22} - a_{21}}{a_{11} + a_{22} - (a_{12} + a_{21})} = \frac{2-(-2)}{2+2-(-2-2)} = \frac{1}{2},\ p_2 = 1 - p_1 = \frac{1}{2};$$

and
$$q_1 = \frac{a_{22} - a_{12}}{a_{11} + a_{12} - (a_{12} + a_{21})} = \frac{2-(-2)}{2+2-(-2-2)} = \frac{1}{2};\ q_2 = 1 - q_1 = \frac{1}{2}.$$

The expected value of game (corresponding to above strategies) is give by

$$V = \frac{a_{11}a_{22} - a_{21}a_{12}}{a_{11} + a_{22} - (a_{12} + a_{21})} = \frac{2\times 2-(-2)\times(-2)}{2+2-(-2-2)} = 0$$

Hence, the optimum strategies for the two players are: *H* : 1/2, *T* : 1/2 with $V = 0$.

Example 6 Players *A* and *B* each take out one or two matches and guess how many matches opponent has taken. If one of the players guesses correctly, then the loser has to pay him as many rupees as the sum of the number held by both players. Otherwise, the payout is zero. Write down the pay-off matrix and obtain the optimal strategies of both players.

Solution The pay-off matrix for the two players is given by

	Player *B*	
Player *A*	1	2
1	2	0
2	0	4

The pay-off matrix does not have any saddle point. The optimum mixed strategies for the two players are:

$$p_1 = \frac{4-0}{2+4-(0+0)} = \frac{2}{3};\ p_2 = 1 - p_1 = 1 - p_1 = \frac{1}{3};.$$

$$q_1 = \frac{4-0}{2+4-(0+0)} = \frac{2}{3};\ q_2 = 1 - q_1 = 1 - q_1 = \frac{1}{3}.$$

The expected value of the game (corresponding to the above strategies) is given by

$$V = \frac{8-0}{2+4-0} = \frac{4}{3}$$

Hence, the optimum strategies for the two players are; Player $A : p_1 = 2/3, p_2 = 1/3$; Player $B : q_1 = 2/3$, $q_2 = 1/3$ with $V = 4/3$.

Example 7 Consider a modified form of "matching biased coins" game problem. The matching player is paid Rs 8.00 if the two coins turn both heads and Re 1.00 if the coins turn both tails. The non-matching player is paid Rs 3.00 when the two coins do not match. Given the choice of being the matching or non-matching player, which one would you choose and what would be your strategy?

[*Delhi Univ., MBA, 1999*]

Solution The pay-off matrix for the matching player is given by,

Matching Player	Non-matching Player *H*	*T*
H	8	–3
T	–3	1

The pay-off matrix has no saddle point. The optimum mixed strategies for the two players are determined by·

$$p_1 = \frac{1-(-3)}{8+1-(-3-3)} = \frac{4}{15};\ p_2 = 1 - p_1 = \frac{11}{15};$$

$$q_1 = \frac{1-(-3)}{8+1-(-3-3)} = \frac{4}{15};\ q_2 = 1 - q_1 = \frac{11}{15};$$

The expected value of the game (corresponding to the above strategies) is given by,

$$V = \frac{8-(-3)(-3)}{8+1-(-3-3)} = -\frac{1}{15}.$$

Hence, the optimum strategies for the Matching players are same as for Non-matching player, i.e. $H : 4/15$; $T = 11/15$ with $V = 1/15$. We would like to be non-matching player.

Example 8 Players *A* and *B* play a game in which each has three coins, a 5*p*, 10*p* and a 20*p*. Each selects a coin without the knowledge of the other's choice. If the sum of the coins is an odd amount, then *A* wins *B*'s coin. But, if the sum is even, then *B* wins *A*'s coin. Find the best strategy for each player and the values of the game.

[*Rajasthan Univ., MBA, 1989; Agra Univ., MCA, 1995*]

Solution The pay-off matrix for player *A* is given by

Player *A*	Player *B* 5*p* : B_1	10*p* : B_2	20*p* : B_3
5*p* : A_1	– 5	10	20
10*p* : A_2	5	– 10	– 10
20*p* : A_3	5	– 20	– 20

The pay-off matrix has no saddle point. While we try to reduce the size of the given pay-off matrix, it may be noted that every element of column B_3 (strategy B_3 for player *B*) is more than or equal to every corresponding element of row B_2 (strategy B_2 for player B). Evidently, the choice of strategy B_3 by the player B will always result in more losses as compared to that of selecting the strategy B_2. Thus, strategy B_3 is inferior to B_2. Hence, delete the B_3 strategy from the pay-off matrix. The reduced pay-off matrix is shown below:

Player A \ Player B	B_1	B_2	B_3
A_1	–5	10	20
A_2	5	– 10	– 10
A_3	5	– 20	– 20

After column B_3 is deleted, it may be noted that strategy A_2 of player A is dominated by his A_3 strategy, since the profit due to strategy A_2 is greater than or equal to the profit due to strategy A_3, regardless of which strategy player B selects. Hence, strategy A_3 (row 3) can be deleted from further consideration. Thus, the reduced pay-off matrix is:

Player A \ Player B	B_1	B_2	Row minimum
A_1	– 5	10	– 5
A_2	5	–10	–10
Column maximum	5	10	

This matrix also has no saddle point. Thus solution to this game can be obtained by applying any of the methods used for mixed-strategy games as discussed later. The optimal strategies for two players are A: 1/2, 1/2 and B: 2/3, 1/3 with $V = 0$.

Example 9 Solve the game whose pay-off matrix is given below:

Player A \ Player B	B_1	B_2	B_3	B_4
A_1	3	2	4	0
A_2	3	4	2	4
A_3	4	2	4	0
A_4	0	4	0	8

[Meerut Univ., MSc(Maths) 1985, 88]

Solution The pay-off matrix has no saddle point. Reducing the size of the given pay-off matrix by using dominance principles.

From player A's point of view, first row is dominated by the third row yielding the reduced 3×4 pay-off matrix. In the reduced matrix from player B's point of view, first column is dominated by the third column. Thus, by deleting the first row and then the first column, the reduced pay-off matrix so obtained is

Player A \ Player B	B_2	B_3	B_4
A_2	4	2	4
A_3	2	4	0
A_4	4	0	8

Now it may be noted that none of the pure strategies of players A and B is inferior to any of their other strategies. However, the average of payoffs due to strategies B_3 and B_4, {(2 + 4)/2; (4 + 0)/2; (0 + 8)/2} = (3, 2, 4) is superior to the pay-off due to strategy B_2 of player B. Thus, strategy B_2 may be deleted from the matrix. The new matrix so obtained is:

Player A \ Player B	B_3	B_4
A_2	2	4
A_3	4	0
A_4	0	8

Again in the reduced matrix, the average of the pay-offs due to strategies A_3 and A_4 of player A, i.e. {(4 + 0)/2; (0 + 8)/2} = (2, 4) is the same as the pay-off due to strategy A_2. Therefore, the player A will gain the same amount even if the strategy A_2 is never used. Hence, after deleting the strategy A_2 from the reduced matrix, a new reduced 2 × 2 pay-off is obtained,

Player A \ Player B	B_3	B_4
A_3	4	0
A_4	0	8

This game has no saddle point. Let player A chooses his strategies A_3 and A_4 with probability p_1 and p_2, respectively such that $p_1 + p_2 = 1$. Also let player B choose his strategies with probability q_1 and q_2, respectively such that $q_1 + q_2 = 1$. Since both players want to retain their interests unchanged, therefore, we may write:

$$4p_1 + 0.p_2 = 0.p_1 + 8p_2$$

or $$4p_1 = 8(1-p_1) \text{ i.e. } p_1 = 2/3$$

$$4q_1 + 0.q_2 = 0.q_1 + 8q_2$$

or $$4q_1 = 8(1-q_1) \text{ i.e. } q_1 = 2/3$$

The optimal strategies of player A and player B are (0, 0, 2/3, 1/3) and (0, 0, 2/3, 1/3), respectively. The value of the game can be obtained by putting value of p_1 or q_1 in either of the expected pay-off equations. That is,

Expected gain to A : $4p_1 + 0.p_2 = 4(2/3) = 8/3$

Expected loss to B : $4q_1 + 0q_2 = 4(2/3) = 8/3$

Example 10 In a game of matching coins with two players, suppose A wins one unit of value when there are two heads, wins nothing when there are two tails and losses 1/2 unit of value when there is one head and one tail. Determine the pay-off matrix, the best strategies for each player and the value of the game to A.

Solution The pay-off matrix for the given matching coin games is given by

Player A \ Player B	B_1	B_2
A_1	1	– 1/2
A_2	– 1/2	0

As the pay-off matrix does not have a saddle point, the game will be solved by algebraic method.

For Player A: Let p_1 and p_2 be probabilities of selecting strategy A_1 and A_2, respectively. Then expected gain to player A when player B uses its B_1 and B_2 strategies, respectively is given by

$$p_1 - (1/2)\,p_2 \geq V \quad ; B \text{ selects } B_1 \text{ strategy} \quad (1)$$

$$-(1/2)\,.\,p_2 + 0\,.\,p_2 \geq V \quad ; B \text{ selects } B_2 \text{ strategy} \quad (2)$$

where

$$p_1 + p_2 = 1 \quad (3)$$

For obtaining value of p_1 and p_2, considering inequalities (1) and (2) as equations and then with the help of Eq. (3), we get $p_1 = -2V$ and $p_2 = -6V$. Substituting these values of p_1 and p_2 in Eq. (3) we get $V = -1/8$. Thus, $p_1 = 0.25$ and $p_2 = 0.75$.

For Player B: Let q_1 and q_2 be the probabilities of selecting strategies B_1 and B_2, respectively. Then the expected loss to player B when player A uses its A_1 and A_2 strategies, respectively is given by

$$q_1 - (1/2)\,.\,q_2 \leq V \quad ; A \text{ selects } A_1 \text{ strategy} \quad (4)$$

$$-(1/2)\,.\,q_1 + 0\,.\,q_2 \leq V \quad ; A \text{ selects } A_2 \text{ strategy} \quad (5)$$

and

$$q_1 + q_2 = 1 \quad (6)$$

Consider inequalities (4) and (5) as equations and then with the help of Eq. (6), we get $q_1 = 2V$ and $q_2 = -6V$. Substituting values of q_1 and q_2 in Eq. (6), we get $V = -1/8$. Thus, $q_1 = 0.25$ and $q_2 = 0.75$. Hence, the probability of selecting strategies optimally for players A and B are (0.25, 0.75) and (0.25, 0.75), respectively and the value of the game is $V = -1/8$.

Example 11 In a small town, there are only two stores, ABC and XYZ that handle sundry goods. The total number of customers is equally divided between the two, because price and quality of goods sold are equal. Both stores have good reputation in the community, and they render equally good customer service. Assume that a gain of customers by ABC is a loss to XYZ and vice versa. Both stores plan to run annual pre-Diwali sales during the first week of November. Sales are advertised through a local newspaper, radio and television media. With the aid of an advertising firm store ABC constructed the game matrix given below. (Figures in the matrix represent a gain or loss of customers).

	Strategy of XYZ		
Strategy of ABC	*Newspaper*	*Radio*	*Television*
Newspaper	30	40	−80
Radio	0	15	−20
Television	90	20	50

Determine optimal strategies and the worth of such strategies for both ABC and XYZ.

[*ICWA, Dec.1987; Jammu Univ., MBA, 1990; AIMA (Dip. in Mgt.), Dec. 1996; Delhi Univ. MBA, 1999, 2001*]

Solution There is no saddle point in the pay-off matrix. Then reducing this matrix by rules of dominance. Since each element in first column is more than the corresponding element in the third column, therefore removing first column from the pay-off matrix, we get

	XYZ	
ABC	B_2	B_3
A_1	40	−80
A_2	15	−20
A_3	20	50

In the reduced pay-off matrix, each element in second row is less than the corresponding element in third row. Thus, deleting second row from the reduced matrix, we get the further reduced 2 × 2 pay-off matrix as shown below:

	XYZ		
ABC	B_2	B_3	Probability
A_1	40	– 80	p_1
A_3	20	50	p_2
Probability	q_1	q_2	

The reduced 2 × 2 pay-off matrix also does not have the saddle point. Thus, both the stores use mixed strategies.

For Store ABC: Let p_1 and p_2 be probabilities of selecting strategy A_1 (newspaper) and A_3 (television), respectively. Then expected gain to store ABC when store XYZ uses its B_2 and B_3 strategies is given by

$$40p_1 + 20p_2 \quad \text{and} \quad -80p_1 + 50p_2; \quad p_1 + p_2 = 1.$$

For store ABC the probability p_1 and p_2 should be such that expected gains under both conditions are equal. That is

$$\begin{aligned} 40p_1 + 20p_2 &= -80p_1 + 50p_2 \\ 40p_1 + 20(1 - p_1) &= -80p_1 + 50(1 - p_1); \; p_1 + p_2 = 1 \\ 150p_1 &= 30 \text{ or } p_1 = 1/5, \text{ and } p_2 = 1 - p_1 = 4/5 \end{aligned}$$

Thus store ABC should apply strategy A_1 (newspaper) with a probability of 1/5 and strategy A_3 (television), with a probability of 4/5.

For Store XYZ: Let q_1 and q_2 be the probabilities of selecting strategy B_2 (radio) and B_3 (television), respectively. Then, the expected loss to store XYZ when store ABC uses its strategies A_1 and A_3 should be

$$\begin{aligned} 40q_1 - 80q_2 &= 20q_1 + 50q_2; \; q_1 + q_2 = 1 \\ 40q_1 - 80(1 - q_1) &= 20q_1 + 50(1 - q_1) \\ 150q_1 &= 130 \text{ or } q_1 = 13/15, \text{ and } q_2 = 1 - q_1 = 2/15 \end{aligned}$$

Thus, store XYZ should apply strategy B_2 (radio) with a probability of 13/15 and strategy B_3 (television) with a probability of 2/15.

Substituting the values of p_1, p_2 or q_1, q_2 in any of the gain or loss equations, we shall get the expected value of the game (i.e. 24) as shown below:

Expected Gain to Store ABC

(i) $40p_1 + 20p_2 = 40 \times (1/5) + 20 \times (4/5) = 24$

(ii) $-80p_1 + 50p_2 = -80 \times (1/5) + 50 \times (4/5) = 24$

Expected Loss to Store XYZ

(i) $40q_1 - 80q_2 = 40 \times (13/15) - 80 \times (2/15) = 24$

(ii) $20q_1 + 50q_2 = 20 \times (13/15) + 20 \times (2/15) = 24$

Here, it may be noted that the expected loss to one store is the same as the expected gain to another store.

Example 12 Two breakfast food manufacturers, ABC and XYZ are competing for an increased market share. The pay-off matrix, shown in the following table, describes the increase in market share for ABC and decrease in market share of XYZ.

ABC	XYZ: *Give Coupons*	*Decrease Price*	*Maintain Present Strategy*	*Increase Advertising*
Give Coupons	2	−2	4	1
Decrease Price	6	1	12	3
Maintain Present Strategy	−3	2	0	6
Increase Advertising	2	−3	7	1

Determine optimal strategies for both the manufacturers and the value of the game.

[*Delhi Univ., MBA, 1990, 95*]

Solution There is no saddle point in the pay-off matrix. Thus reducing the size of the pay-off marix by rules of dominance. Each element of first row is less than the corresponding elements of second row, therefore deleting first row. The reduced matrix becomes as shown below:

ABC	XYZ: B_1	B_2	B_3	B_4
A_2	6	1	12	3
A_3	−3	2	0	6
A_4	2	−3	7	1

In the reduced matrix, each element of fourth column is more than the corresponding element in second column. Thus, after deleting fourth column the reduced matrix becomes

ABC	XYZ: B_1	B_2	B_3
A_2	6	1	12
A_3	−3	2	0
A_4	2	−3	7

Further compare rows 1 and 3 and then columns 1 and 3 and delete the less attractive row and column from ABC's and XYZ's point of view. The reduced pay-off matrix is shown below:

ABC	XYZ: *Give Coupons* B_1	*Decrease Price* B_2	*Probability*
Decrease Price, A_2	6	1	p_1
Maintain Present Strategy, A_3	−3	2	p_2
Probability	q_1	q_2	

The reduced 2 × 2 pay-off matrix also does not have the saddle point. Thus, both ABC and XYZ use mixed strategies.

For ABC: Let p_1 and p_2 be probabilities of selecting strategy A_2 (decrease price) and A_3 (maintain present strategy), respectively. Then the expected gain to ABC when XYZ uses its B_1 and B_2 strategies is given by

$$6p_1 - 3p_2 \quad \text{and} \quad p_1 + 2\,p_2\,; \quad p_1 + p_2 = 1$$

The probability p_1 and p_2 should be such that expected gains under both conditions are equal. That is,

$$6p_1 - 3p_2 = p_1 + 2p_2$$

$$6p_1 - 3(1 - p_1) = p_1 + 2(1 - p_1)\,;\, p_1 + p_2 = 1$$

$$10p_1 = 5 \text{ or } p_1 = 1/2 \quad \text{and} \quad p_2 = 1 - p_1 = 1/2$$

Hence, ABC should adopt strategy A_2 (decrease price) 50 per cent of time and strategy A_3 (maintain present strategy) 50 per cent of time.

For XYZ: Let q_1 and q_2 be probabilities of selecting strategies B_1 (give coupons) and B_2 (decrease price), respectively. Then the expected loss to XYZ when ABC uses its A_2 and A_3 strategies should be

$$6q_1 + q_2 = -3q_1 + 2q_2 \;;\; q_1 + q_2 = 1$$
$$6q_1 + (1 - q_1) = -3q_1 + 2(1 - q_1)$$
$$10q_1 = 1 \text{ or } q_1 = 1/10 \quad \text{and} \quad q_2 = 1 - q_1 = 9/10$$

Hence, XYZ should adopt strategy B_1 (give coupons) 10 per cent of time and strategy B_2 (decrease price), 90 per cent of time.

The expected gain and loss to ABC and XYZ can be calculated as shown below:

$$\text{Expected gain to ABC}: \quad 6p_1 - 3p_2 = 6(1/2) - 3(1/2) = 3/2$$
$$p_1 + 2p_2 = 1/2 + 2(1/2) = 3/2$$
$$\text{Expected loss to XYZ}: \quad 6q_1 + q_2 = 6(1/10) + (9/10) = 3/2$$
$$-3q_1 + 2q_2 = -3(1/10) + 2(9/10) = 3/2$$

Example 13 Two Firms A and B have for years been selling a competing product which forms a part of both firms' total sales. The marketing executive of Firm A raised the question: 'What should be the firm's strategies in terms of advertising for the product in question'. The market research team of Firm A developed the following data for varying degree of advertising:

(i) No advertising, medium advertising, and large advertising for both firms will result in equal market shares.

(ii) Firm *A* with no advertising: 40 per cent of the market with medium advertising by Firm *B* and 28 per cent of the market with large advertising by Firm *B*.

(iii) Firm *A* using medium advertising: 70 per cent of the market with no advertising by Firm *B* and 45 per cent of the market with large advertising by Firm *B*.

(iv) Firm *A* using large advertising: 75 per cent of the market with no advertising by Firm *B* and 47.5 per cent of the market with medium advertising by Firm *B*.

(a) Based upon the foregoing information, answer the marketing executive's questions:

[*Delhi Univ., MBA, 1998, 2000; AIMA (Dip. in Mgt.), 1989; Sardar Patel Univ., MBA, 1997*]

(b) What advertising policy should Firm *A* pursue when consideration is given to the above factors: selling price, Rs 4 per unit; variable cost of product, Rs 2.50 per unit; annual volume of 30,000 units for Firm *A*; cost of annual medium advertising Rs 5,000 and cost of annual large advertising Rs 15,000? What contribution, before other fixed costs, is available to the firm?

[*AIMA (Dip. in Mgt.), June 1988; Delhi Univ., MBA, 1998, 2000*]

Solution The pay-off matrix of the game between Firms *A* and *B* is as follows:

Firm *A*	Firm *B*: *No Advt.*, B_1	*Medium Advt.*, B_2	*Large Advt.*, B_3	Row minimum
No Advt., A_1	50	40	28	28
Medium Advt., A_2	70	50	45	45
Large Advt., A_3	75	47.5	50	47.5
Column maximum	75	50	50	

From the pay-off matrix it is observed that there is no saddle point in the problem.

Applying rules of dominance, delete first row (dominated by third row) and then first column (dominated by both columns 2 and 3) from the pay-off matrix. The reduced pay-off matrix so obtained is shown below:

Firm A \ Firm B	B_2	B_3	*Probability*
A_2	50	45	p_1
A_3	47.5	50	p_2
Probability	q_1	q_2	

The reduced 2 × 2 pay-off matrix also does not have the saddle point. Thus, both the firms use mixed strategies. Adopting the same procedure as discussed in earlier examples, the expected gain to Firm *A* can be calculated as follows:

Expected Gain to Firm A

$$50p_1 + 47.5p_2 = 45p_1 + 50p_2 \; ; \quad p_1 + p_2 = 1$$

$$50p_1 + 47.5(1 - p_1) = 45p_1 + 50(1 - p_1)$$

$$7.5p_1 = 2.5 \quad \text{or} \quad p_1 = 1/3 \quad \text{and} \quad p_2 = 1 - p_1 = 2/3.$$

$$\text{Expected gain} = 50p_1 + 47.5p_2 = 50(1/3) + 47.5(2/3) = 145/3.$$

Thus, the optimal policy for Firm *A* is to apply strategy A_2 (medium advertising) with probability 0.33 and strategy A_3 (large advertising) with probability 0.67 on any one play of the game. With this policy, the firm may expect to gain 145/3 = 48.3 per cent of the market share.

Market Share of Firm A

Firm A \ Firm B	*No Advt.*	*Medium Advt.*	*Large Advt.*
No Advt.	0.50 × 30,000 = 15,000	0.40 × 30,000 = 12,000	0.28 × 30,000 = 8,400
Medium Advt.	0.70 × 30,000 = 21,000	0.50 × 30,000 = 15,000	0.45 × 30,000 = 13,500
Large Advt.	0.75 × 30,000 = 22,500	0.475 × 30,000 = 14,250	0.50 × 30,000 = 15,000

Given that the expenditure on medium and large advertisements is Rs 5,000 and Rs 15,000, respectively, net profit to Firm *A* can be calculated by using following equation:

$$\text{Net profit} = (\text{Sales price} - \text{Cost price}) \times \text{Sales volume} - \text{Advertising expenditure}$$

The net profit to Firm *A* is shown below:

Profit to Firm A

Firm A \ Firm B	*No Advt.*	*Medium Advt.*	*Large Advt.*
No advt.	1.5 × 15,000 = 22,500	1.5 × 12,000 = 18,000	1.5 × 8,400 = 12,60
Medium advt.	1.5 × 21,000 – 5,000 = 26,500	1.5 × 15,000 – 5,000 = 17,500	1.5 × 13,500 – 5,000 = 15,250
Large advt.	1.5 × 22,500 – 15,000 = 18,750	1.5 × 14,250 – 15,000 = 6,375	1.5 × 15,000 – 15,000 = 7,500

Observations

1. If Firm A chooses the strategy of '*No advertising*', then minimum profit is Rs 12,600, because Firm B can adopt its strategy '*Large advertising*'.
2. If Firm A chooses the strategy of '*Medium advertising*', then minimum profit is Rs 15,250 because Firm B can again adopt its strategy '*Large advertising*'.
3. If Firm A chooses the strategy of '*Large advertising*', then minimum profit is Rs 6,375 because Firm B can adopt its strategy '*Medium advertising*'.

Based on these observations, the Firm A must adopt the policy of 'medium advertising' to gain maximum profit of Rs 15,250 among these three alternatives, and must spend Rs 5,000 for advertising.

Example 14 Two competitors are competing for the market share of the similar product. The pay-off matrix in terms of their advertising plan is shown below:

	Competitor B		
Competitor A	*No Advertising*	*Medium Advertising*	*Heavy Advertising*
No Advertising	10	5	–2
Medium Advertising	13	12	13
Heavy Advertising	16	14	10

Suggest optimal strategies for the two firms and the net outcome thereof.

[*Delhi Univ., MCom, 1990; MBA, 1994; HP Univ., MBA, 1999*]

Solution Applying rules of dominance to delete first column (dominated by second column) and then first row (dominated by second as well as third rows) from the pay-off matrix, the reduced pay-off matrix so obtained is shown below:

	Firm B	
Firm A	*Medium Advt.* B_2	*Heavy Advt.* B_3
Medium Advt. A_2	12	15
Heavy Advt. A_3	14	10

As the pay-off matrix does not have saddle point, firms will use mixed strategies. Applying arithmetic method to get optimal mixed strategies for both the firms, the results are:

	Firm B		
Firm A	B_2	B_3	
A_2	12	15	$14 - 10 = 4,\ p(A_2) = \frac{4}{4+3} = \frac{4}{7}$
A_3	14	10	$15 - 12 = 3,\ p(A_3) = \frac{3}{4+3} = \frac{3}{7}$
	$15 - 10 = 5$	$14 - 12 = 2$	

$$p(B_2) = \frac{5}{5+2} = \frac{5}{7} \qquad p(B_3) = \frac{2}{5+2} = \frac{2}{7}$$

Hence, Firm A should adopt strategy A_2 and A_3, 57 per cent of the time and 43 per cent of time, respectively (or with 57 per cent and 43 per cent probability on any one play of the game, respectively). Similarly, Firm B should adopt strategy B_2 and B_3, 71 per cent of time and 29 per cent of time, respectively (or with 71 per cent and 29 per cent probability on any one play of the game, respectively).

Expected Gain to Firm A

(i) $12 \times (4/7) + 14 \times (3/7) = 90/7$, Firm B adopt B_2
(ii) $15 \times (4/7) + 10 \times (3/7) = 90/7$, Firm B adopt B_3

Expected Loss to Firm B

(i) $12 \times (5/7) + 15 \times (2/7) = 90/7$, Firm A adopt A_2
(ii) $14 \times (5/7) + 10 \times (2/7) = 90/7$, Firm A adpot A_3

Example 15 Solve the following game after reducing it to a 2×2 game

	Player B		
Player A	B_1	B_2	B_3
A_1	1	7	2
A_2	6	2	7
A_3	5	1	6

[*Osmania Univ., MSc (Maths), 1985; Meerut Univ., MSc (Maths), 1986*]

Solution In the given game matrix, the third row is dominated by second row and in the reduced matrix third column is dominated by the first column. So after elimination of the third row and the third column the game matrix becomes

	Player B	
Player A	B_1	B_2
A_1	1	7
A_2	6	2

The optimal strategy mix for player A is: $p_1 = 4/10 = 2/5$ and $p_2 = 6/10 = 3/5$, where p_1 and p_2 represent the probabilities of player A's using his strategies A_1 and A_2, respectively.

The optimal strategy mixture for player B is: $q_1 = 5/10 = 1/2$ and $q_2 = 5/10 = 1/2$, where q_1 and q_2 represent the probabilities of player B's using his strategies B_1 and B_2, respectively. The value of the game is 4.

Example 16 Use graphical method in solving the following game and find the value of the game.

	Player B			
Player A	B_1	B_2	B_3	B_4
A_1	2	2	3	−2
A_2	4	3	2	6

[*Madras Univ., MBA, 1996*]

Solution The game does not have a saddle point. If the probability of player A's playing A_1 and A_2 in the strategy mixture is denoted by p_1 and p_2, respectively, where $p_2 = 1 - p_1$, then the expected pay-off (gain) to player A will be

B's Pure Strategies	*A's Expected Pay-off*
B_1	$2p_1 + 4p_2$
B_2	$2p_1 + 3p_2$
B_3	$3p_1 + 2p_2$
B_4	$-2p_1 + 6p_2$

These four expected pay-off lines can be plotted on the graph to solve the game.

The graph for player A: A graphic solution is shown in Fig. 2. Here, the probability of player A's playing A_1, i.e. p_1 is measured on the x-axis. Since p_1 cannot exceed 1, the x-axis is cut-off at $p_1 = 1$. The expected pay-off of player A is measured along y-axis. From the game matrix, if player B plays B_1, the expected pay-off of player A is 2 when A plays A_1 with $p_1 = 1$ and 4 when A plays A_2 with $p_1 = 0$. These two extreme points are connected by a straight line, which shows the expected pay-off of A when B plays B_1. Three other straight lines are similarly drawn for B_2, B_3 and B_4.

It is assumed that player B will always play his best possible strategies yielding the worst result to player A. Thus, the payoffs (gains) to A are represented by the lower boundary when he is faced with the most unfavourable situation in the game. Since player A must choose his best possible strategies in order to realize a maximum expected gain, the highest expected gain is found at point P, where two straight lines

$$E_3 = 3p_1 + 2p_2 = 3p_1 + 2(1 - p_1) \quad \text{and} \quad E_4 = -2p_1 + 6p_2 = -2p_1 + 6(1 - p_1)$$

meet. In this manner the solution to the original (2 × 4) game reduces to that of the game with pay-off matrix of size (2 × 2) as given below:

	Player B	
Player A	B_3	B_4
A_1	3	−2
A_2	2	6

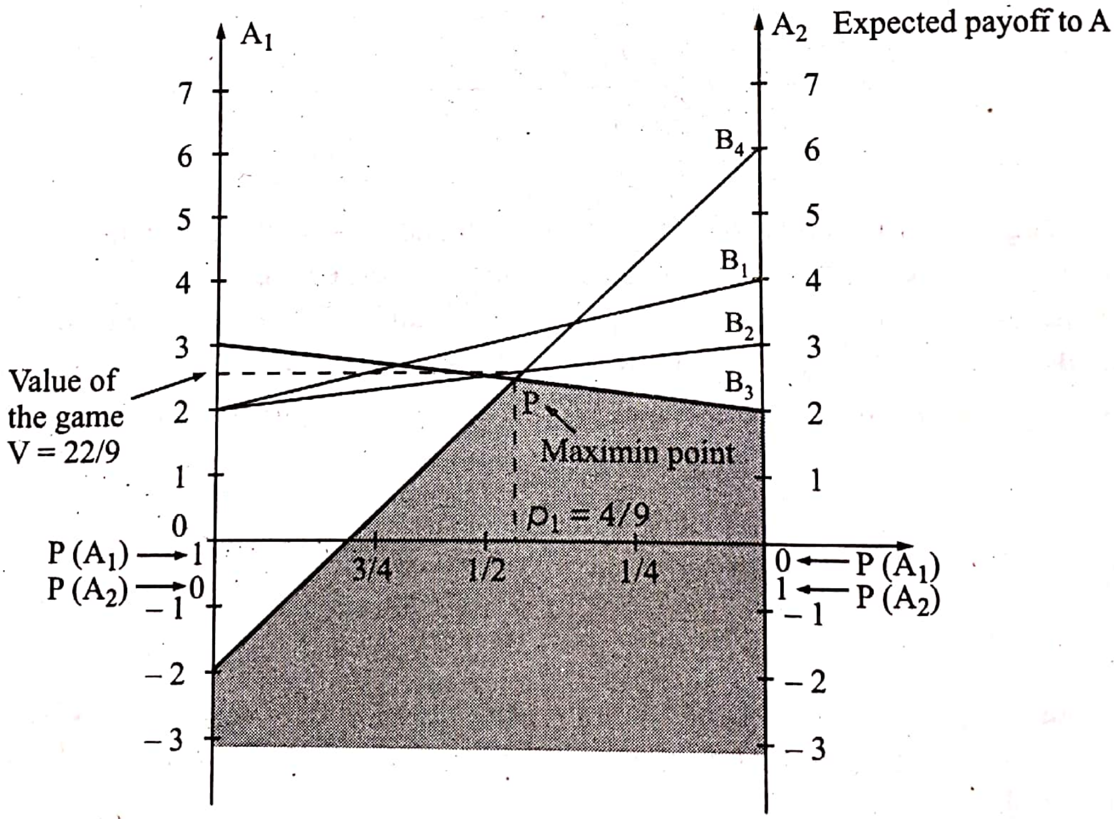


Fig. 2 Graph for Player A

The optimum pay-off to player A can now be obtained by setting E_3 and E_4 equal and solving for p_1, i.e.

$$3p_1 + 2(1 - p_1) = -2p_1 + 6(1 - p_1) \text{ or } p_1 = 4/9; \quad p_2 = 1 - p_1 = 5/9$$

Substituting the value of p_1 and p_2 in the equation for E_3 (or E_4) we have,

$$\text{Value of the game, } V = 3 \times 4/9 + 2 \times 5/9 = 22/9$$

The optimal strategy mix of player B can also be found in the same manner as for player A. If the probabilities of B's selecting strategy B_3 and B_4 are denoted by q_3 and q_4, respectively, then the expected loss to B will be

$$L_3 = 3q_3 - 2q_4 = 3q_3 - 2(1 - q_3) \text{ (if } A \text{ selects } A_1)$$
$$L_4 = 2q_3 + 6q_4 = 2q_3 + 6(1 - q_3) \text{ (if } A \text{ selects } A_2)$$

To solve for q_3, equate the two equations

$$3q_3 - 2(1 - q_3) = 2q_3 + 6(1 - q_3) \text{ or } q_3 = 8/9;\ q_4 = 1 - q_3 = 1/9$$

Substituting the value of q_3 and q_4 in the equation for L_3 (or L_4), we have

$$\text{Value of the game, } V = 3 \times 8/9 - 2 \times 1/9 = 22/9$$

Example 17 Two firms A and B make colour and black & white television sets. Firm A can make either 150 colour sets in a week or an equal number of black & white sets, and make a profit of Rs 400 per colour set, or 150 colour and 150 black & white sets, or 300 black & white sets per week. It also has the same profit margin on the two sets as A. Each week there is a market of 150 colour sets and 300 black & white sets and the manufacturers would share market in the proportion in which they manufacture a particular type of set.

Write the pay-off matrix of A per week. Obtain graphically A's and B's optimum strategies and value of the game. *[Bombay Univ., MMS, 1997]*

Solution For firm A, the strategies are:

A_1 : make 150 colour sets, A_2 : make 150 black & white sets.

For firm B, the strategies are:

B_1 : make 300 colour sets, B_2 : make 150 colour and 150 black & white sets.

B_3 : make 300 black and white sets.

For the combination A_1B_1, the profit to firm A would be: $\{150/(150 + 300)\} \times 150 \times 400$ = Rs 20,000 wherein $150/(150 + 300)$ represents share of market for A, 150 is the total market for colour television sets and 400 is the profit per set. In a similar manner, other profit figures may be obtained as shown in the following pay-off matrix:

	B's Strategy		
A's Strategy	B_1	B_2	B_3
A_1	20,000	30,000	60,000
A_2	45,000	45,000	30,000

This pay-off table has no saddle point. Thus to determine optimum mixed strategy, the data are plotted on graph as shown in Fig. 3.

Lines joining the pay-offs on axis A_1 with the pay-offs on axis A_2 represents each of B's strategies. Since firm A wishes to maximize his minimum expected pay-off, we consider the highest point of intersection, P on the lower envelope of A's expected pay-off equation. This point P represents the maximum expected value of the game. The lines B_1 and B_3 passing through P, define the strategies which firm B needs to adopt. The solution to the original 2×3 game, therefore, reduces to that of the simpler game with 2×2 pay-off matrix as follows:

	B's Strategy		
A's Strategy	B_1	B_3	*Probability*
A_1	20,000	60,000	p_1
A_2	45,000	30,000	p_2
Probability	q_1	q_2	

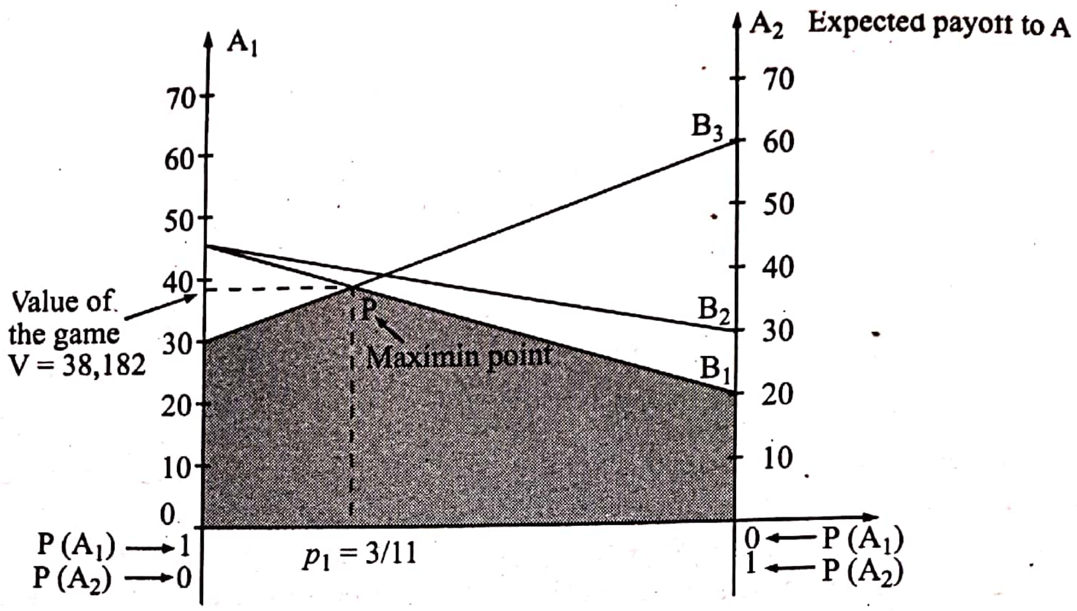


Fig. 3 Graph for Player A

The optimal mixed strategies of player A are: $A_1 = 3/11, A_2 = 8/11$. Similarly, the optimal mixed strategies for B are: $B_1 = 6/11$, $B_2 = 0$, $B_3 = 5/11$. The value of the game is $V = 38{,}182$.

Example 18 Obtain the optimal strategies for both persons and the value of the game for two-person zero-sum game whose pay-off matrix is as follows:

	Player B	
Player A	B_1	B_2
A_1	1	−3
A_2	3	5
A_3	−1	6
A_4	4	1
A_5	2	2
A_6	−5	0

[*Dibrugarh Univ., MSc (Stat), 1994; Karnataka Univ., BE (Ind.), 1994*]

Solution The game does not have any saddle point. If the probability of player B's playing strategies B_1 and B_2 in the strategy mix is denoted by q_1 and q_2 such that $q_1 + q_2 = 1$, then the expected pay-off to player B will be:

A's Pure Strategies	*B's Expected Pay-off*
A_1	$q_1 - 3q_2$
A_2	$3q_1 + 5q_2$
A_3	$-q_1 + 6q_2$
A_4	$4q_1 + q_2$
A_5	$2q_1 + 2q_2$
A_6	$-5q_1 + 0q_2$

The six expected pay-off lines can be plotted on the graph to solve the game.

The graph for player B: A graphic solution is shown in Fig. 4 where the probability of player *B*'s playing B_1, i.e. q_1 is measured on the *x*-axis. Since q_1 cannot exceed 1, therefore *x*-axis is cut off at $q_1 = 1$. The expected pay-off of player *B* is measured along *y*-axis. From the game matrix, if player *A* plays A_1, the expected pay-off of player *B* is 1 when he plays B_1 with $q_1 = 1$ and -3 when he plays B_2 with $q_1 = 0$. These two extreme points are connected by a straight line, which shows the expected pay-off to *B* when *A* plays A_1. Five other straight lines are similarly drawn for A_2 to A_6.

It is assumed that player *A* will always play his best possible strategies yielding the worst result to player *B*. Thus, pay-offs (losses) to *B* are represented by the upper boundary when he is faced with the most unfavourable situation in the game. According to the minimax criterion, player *B* will always select a combination of strategies B_1 and B_2, so that he minimizes the losses. In this case also the optimum solution occurs at the intersection of the two pay-off lines.

$$E_3 = 3q_1 + 5q_2 = 3q_1 + 5(1 - q_1) \text{ and } E_4 = 4q_1 + q_2 = 4q_1 + (1 - q_1)$$

The solution to the original (6 × 2) game reduces to that of the game with pay-off matrix of size (2 × 2) as shown below:

	Player *B*	
Player *A*	B_1	B_2
A_2	3	5
A_4	4	1

Now using the usual method of solution for a (2 × 2) game, the optimum strategies can be obtained by setting E_3 and E_4 equal, i.e. $3q_1 + 5(1 - q_1) = 4q_1 + (1 - q_1)$ or $q_1 = 4/5$ and $q_2 = 1/5$. Similarly, we can obtain optimal mixed strategies for player *A* as: $3p_2 + 4p_4 = 5p_2 + p_4$ to obtain $p_2 = 3/5$ and $p_4 = 2/5$ whereas $p_1 = p_3 = p_5 = p_6 = 0$ and, value of the game, $V = 17/5$.

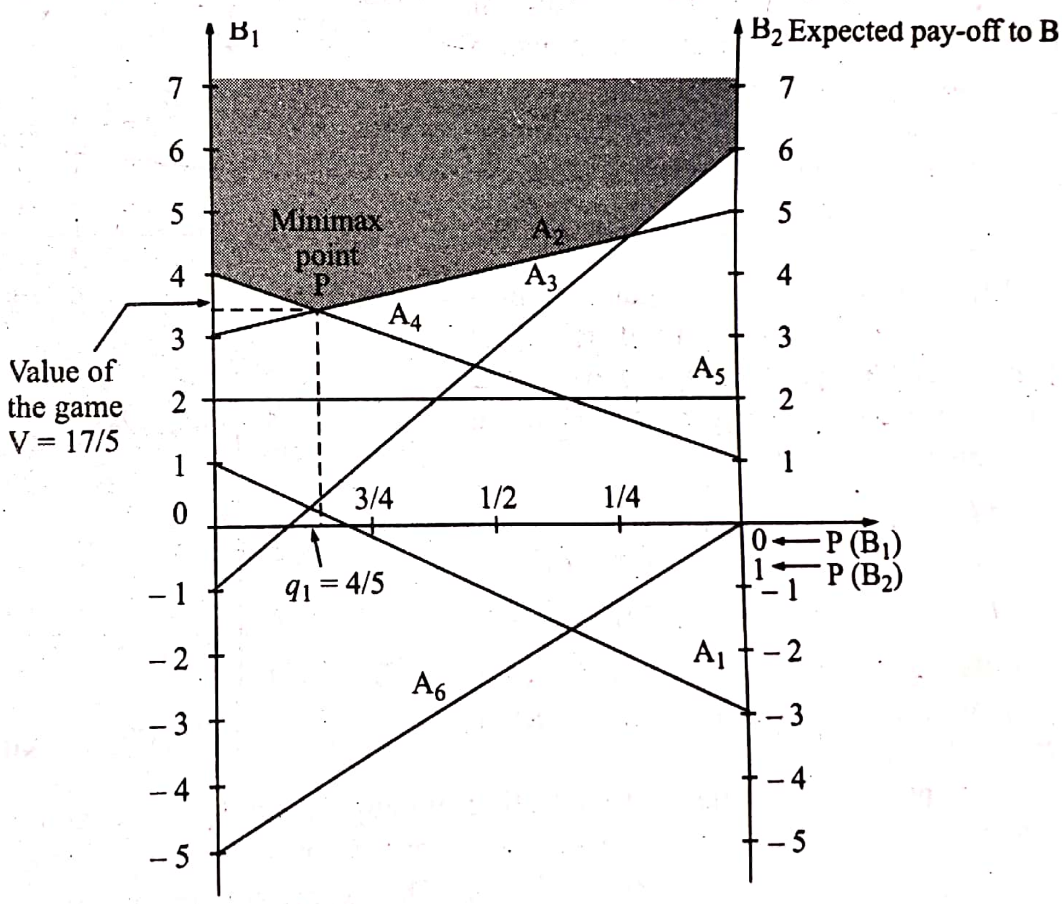


Fig. 4 Graph for Player *B*

Example 19 Solve the following game graphically:

	Player *B*	
Player *A*	B_1	B_2
A_1	1	2
A_2	4	5
A_3	9	–7
A_4	–3	–4
A_5	2	1

[*Madurai Univ., MSc (Maths)*, 198

Solution The given pay-off matrix has no saddle point. So, let the player B play the mixed strategies with probability q_1 and q_2 such that, $q_1 + q_2 = 1$. Then, B's expected pay-offs against A's pure strategies are given by

A's Pure Action	*B's Expected Pay-off* (E_i)
A_1	$q_1 + 2q_2$
A_2	$4q_1 + 5q_2$
A_3	$9q_1 + 7q_2$
A_4	$-3q_1 - 4q_2$
A_5	$2q_1 + q_2$

The graph for player B: The expected pay-off equations are plotted as shown in Fig. 5. The point P represents the minimax expected value of the game for player B. The minimal point occurs at the intersection of two pay-off lines

$$E_2 = 4q_1 + 5q_2 \quad \text{and} \quad E_3 = 9q_1 - 7q_2.$$

The solution to the 5 × 2 game reduces to that of the game with pay-off matrix of size (2 × 2). The optimum pay-off to player B can be obtained by setting E_2 and E_3 equal and solving for q_1, i.e.

$$4q_1 + 5q_2 = 4q_1 - 7q_2$$

or $\quad 4q_1 + 5(1 - q_1) = 9q_1 - 7(1 - q_1)$

or $\quad q_1 = 12/17 \quad$ and $\quad q_2 = 1 - q_1 = 5/17.$

Substituting the value of q_1 and q_2 in equation E_2 (or E_3), we have the expected loss to B as $V = 73/13$.

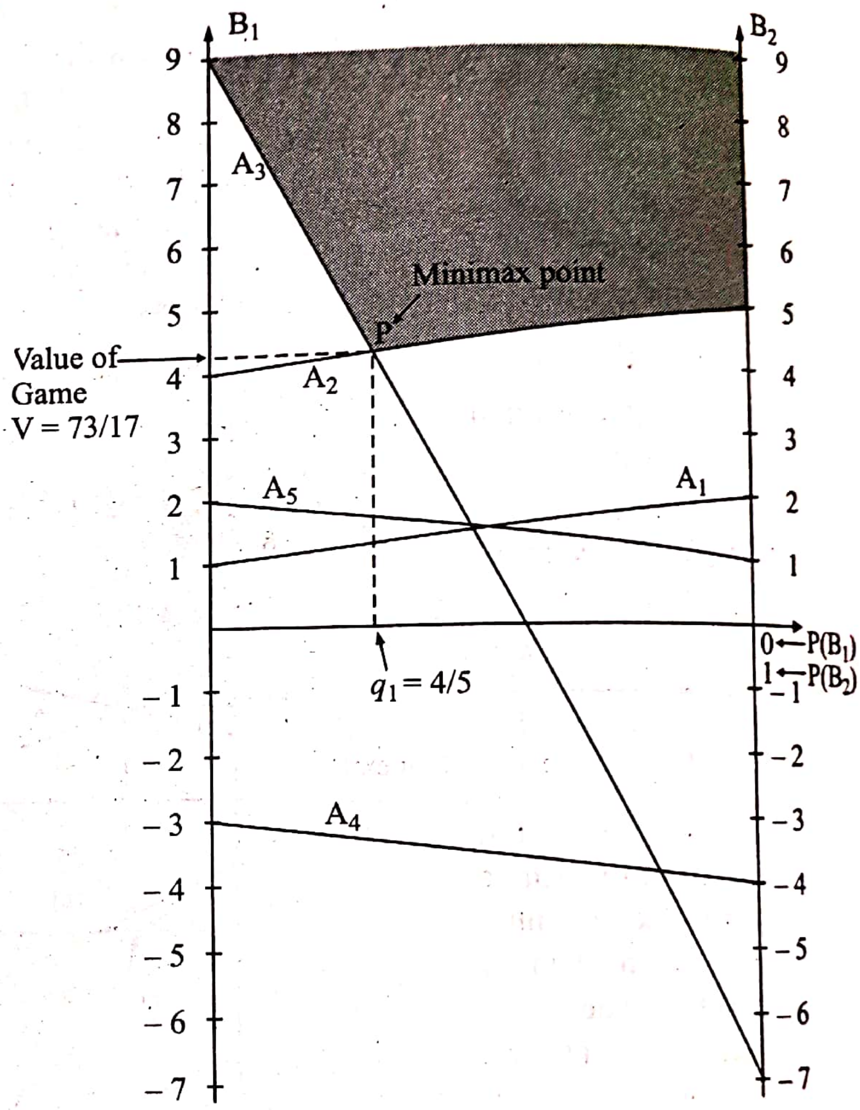


Fig. 5 Graph for Player B

If the probability of player A's selecting strategy A_2 and A_3 are p_2 and p_3, respectively, then the expected gain to A can be calculated by equating two pay-off lines: $4p_2 + 9p_3 = 5p_2 - 7p_3$ to obtain $p_2 = 16/17$, $p_3 = 1 - p_2 = 1/17$ whereas $p_1 = 0$ and $p_4 = 0$. The expected gain to A is $V = 73/17$.

Example 20 For the following pay-off matrix, transform the zero-sum game into an equivalent linear programming problem and solve it by using simplex method.

		Player B	
Player A	B_1	B_2	B_3
A_1	1	–1	3
A_2	3	5	–3
A_3	6	2	–2

Solution The first step is to find out the saddle point (if any) in the pay-off matrix as shown below:

Player A	Player B B_1	B_2	B_3	Row minimum
A_1	1	[–1]	(3)	[–1] ← *Maximin*
A_2	3	5	–3	–3
A_3	6	2	–2	–2
Column maximum	6	5	(3) ← *Minimax*	

The given game pay-off matrix does not have a saddle point. Since the maximin value is –1, therefore, it is possible that the value of game (V) may be negative or zero because $-1 < V < 1$. Thus, a constant which is at least equal to the negative of maximin value, i.e. more than –1 is added to all the elements of the pay-off matrix. Thus, adding a constant number 4 to all the elements of the pay-off matrix, the pay-off matrix becomes:

Player A	Player B B_1	B_2	B_3	*Probability*
A_1	5	3	7	p_1
A_2	7	9	1	p_2
A_3	10	6	2	p_3
Probability	q_1	q_2	q_3	

Let p_i (i = 1, 2, 3) and q_j (j = 1, 2, 3) be the probabilities of selecting strategies A_i (i = 1, 2, 3) and B_j (j = 1, 2, 3) by players A and B, respectively. The expected gain for player A will be as follows:

For strategy

$$B_1 : 5p_1 + 7p_2 + 10p_3 \geq V \text{ or } 5\frac{p_1}{V} + 7\frac{p_2}{V} + 10\frac{p_3}{V} \geq 1$$

$$B_2 : 3p_1 + 9p_2 + 6p_3 \geq V \text{ or } 3\frac{p_1}{V} + 9\frac{p_2}{V} + 6\frac{p_3}{V} \geq 1$$

$$B_3 : 7p_1 + p_2 + 2p_3 \geq V \text{ or } 7\frac{p_1}{V} + \frac{p_2}{V} + 2\frac{p_3}{V} \geq 1$$

$$p_1 + p_2 + p_3 = 1 \text{ or } \frac{p_1}{V} + \frac{p_2}{V} + \frac{p_3}{V} = \frac{1}{V}$$

and

$$p_1, p_2, p_3 \geq 0.$$

In order to simplify, we define new variables: $x_1 = p_1/V$, $x_2 = p_2/V$ and $x_3 = p_3/V$. The problem for player A, therefore becomes,

Minimize Z_p (= $1/V$) = $x_1 + x_2 + x_3$

subject to

$$5x_1 + 7x_2 + 10x_3 \geq 1$$
$$3x_1 + 9x_2 + 6x_3 \geq 1$$
$$7x_1 + x_2 + 2x_3 \geq 1$$

and

$$x_1, x_2, x_3 \geq 0.$$

Player B's objective is to minimize his expected losses which can be reduced to minimizing the value of the game V. Hence, the problem of player B can be expressed as follows:

Maximize Z_q (= $1/V$) = $y_1 + y_2 + y_3$

subject to

$$5y_1 + 3y_2 + 7y_3 \leq 1$$
$$7y_1 + 9y_2 + y_3 \leq 1$$
$$10y_1 + 6y_2 + 2y_3 \leq 1$$

and

$$y_1, y_2, y_3 \geq 0$$

where $y_1 = q_1/V$; $y_2 = q_2/V$ and $y_3 = q_3/V$.

It may be noted that problem of player A is the dual of the problem of player B. Therefore, solution of the dual problem can be obtained from the optimal simplex table of primal.

To solve the problem of player B, introduce slack variables to convert the three inequalities to equalities. The problem becomes

Maximize $Z_q = y_1 + y_2 + y_3 + 0s_1 + 0s_2 + 0s_3$

subject to the constraints

$$5y_1 + 3y_2 + 7y_3 + s_1 = 1$$
$$7y_1 + 9y_2 + y_3 + s_2 = 1$$
$$10y_1 + 6y_2 + 2y_3 + s_3 = 1$$

and $$y_1, y_2, y_3, s_1, s_2, s_3 \geq 0$$

The initial solution is shown om Table 1.

Table 1 Initial Solution

		$c_j \rightarrow$	1	1	1	0	0	0	
Unit Cost c_B	*Variables in Basis* **B**	*Solution Values* y_B (= b)	y_1	y_2	y_3	s_1	s_2	s_3	*Min. Ratio* y_B/y_1
0	s_1	1	5	3	7	1	0	0	1/5
0	s_2	1	7	9	1	0	1	0	1/7
0	s_3	1	(10)	6	2	0	0	1	1/10
$Z = 0$		z_j	0	0	0	0	0	0	
		$c_j - z_j$	1	1	1	0	0	0	

Proceeding with usual simplex method, the optimal solution is shown in Table 2.

Table 2 Optimal Solution

		$c_j \rightarrow$	1	1	1	0	0	0
Unit Cost c_B	*Variables in Basis* **B**	*Solution Values* y_B (= b)	y_1	y_2	y_3	s_1	s_2	s_3
1	y_3	1/10	2/5	0	1	3/20	– 1/10	0
1	y_2	1/10	11/15	1	0	– 1/60	7/60	0
0	s_3	1/5	24/5	0	0	– 1/5	– 3/5	1
Z = 1/5		z_j	17/15	0	0	2/15	1/15	0
		$c_j - z_j$	– 2/15	0	0	– 2/15	– 1/15	0

The optimal solution (mixed strategies) for B is: $y_1 = 0$; $y_2 = 1/10$ and $y_3 = 1/10$ and the expected value of the game is: $Z = 1/V$ – constraint (= 4) = 5 – 4 = 1. These solution values are now converted back into the original variables: If $1/V = 1/5$ then $V = 5$

$$y_1 = q_1/V, \text{ then } q_1 = y_1 \times V = 0$$
$$y_2 = q_2/V, \text{ then } q_2 = y_2 \times V = 1/10 \times 5 = 1/2$$
$$y_3 = q_3/V, \text{ then } q_3 = y_3 \times V = 1/10 \times 5 = 1/2$$

The optimal strategies for player A are obtained from the $c_j - z_j$ row of the Table 2.

$$x_1 = 2/15,\ x_2 = 1/15 \text{ and } x_3 = 0$$

Then $p_1 = x_1 \times V = (2/15) \times 5 = 2/3$; $p_2 = x_2 \times V = (1/15) \times 5 = 1/3$ and $p_3 = x_3 \times V = 0$.

Hence, the probabilities of using strategies by both the players are: Player A = (2/3, 1/3, 0); Player B = (0, 1/2, 1/2) and, value of the game is, $V = 1$.

Example 21 Solve the following game approximately:

	Player B		
	1	−1	−1
Player A	−1	−1	3
	−1	2	−1

[*Sambalpur Univ., MSc (Maths), 1986; Meerut Univ., MSc (Maths), 1988; IIIE (Grad), 1989; Jodhpur Univ., MSc (Maths), 1992; Dibrugarh Univ., MSc (Stat), 1994*]

Solution Let the player A select the second row, being the superior among his other strategies and place it under the matrix. Player B examines this row and chooses first column corresponding to the *smallest* number of this row. First column is then placed to the right of the matrix. Player A examines this column and chooses first row corresponding to the *largest* number in this column. First row is then added to the row last chosen. Player B then chooses the column corresponding to the smallest number in the new row and adds this column to the column last chosen. In case of a tie the player will select that row or column which is different from his last choice. The procedure is repeated for a finite number of iterations. Following ten iterations are presented with the smallest elements in each succeeding row with the largest elements in each succeeding column being encircled.

	Player B													
	1	−1	−1	(1)	0	(1)	0	−1	(0)	−1	(0)	−1	0	4/10
Player A	−1	−1	3	−1	−2	−3	−4	−1	−2	(1)	0	(3)	(2)	3/10
	−1	2	−1	−1	(1)	0	(2)	(1)	0	−1	−2	−3	−4	3/10

(−1)	−1	3
0	(−2)	2
(−1)	0	1
0	(−1)	0
−1	1	(−1)
−2	3	(−2)
−1	2	(−3)
(−2)	1	0
−1	0	(−1)
(−2)	−1	2
4/10	2/10	4/10

The approximate strategies after 10 iterations are found by dividing the number of encircled elements in each row or column by total number of iterations. Thus, optimal mixed strategies for players A and B are:

Player A: A_1 : 4/10, A_2 : 3/10, A_3 : 3/10 and Player B: B_1 : 4/10, B_2 : 2/10, B_3 : 4/10

The upper bound for the value of the game can be determined by dividing the largest element 2, in the last column by the total number of iterations. Likewise, the lower bound can be determined by dividing the smallest element –2, in the last row by the total number of iterations. Thus, the approximate value of the game is $-2/10 \leq V \leq 2/10$, i.e. $-1/5$ $V \leq 1/5$.

Example 22 In a well-known children's game, each player says 'stone' or 'scissors' or 'papers'. If one says 'stone' and the other 'scissors', then the former wins a rupee. Similarly 'scissors' beats 'paper' and 'paper' beats 'stone', i.e. the player calling the former word wins a rupee. If the two players name the same item, then there is a tie, i.e. there is no pay-off. Write down the pay-off matrix. Find the value of the game and hence write down the optimal strategies of both players. *[AIMA, (Dip. in Mgt.), 1989]*

Solution Let A and B play the game. Then the pay-off matrix for player A is given by

	Player B		
Player A	Stone	Paper	Scissors
Stone	0	1	1
Paper	–1	0	1
Scissors	1	1	0

Let the player A select, arbitrarily, the third row as his strategy and place it under the given pay-off matrix. Player B examines this row and chooses second column corresponding to the smallest number of this row. Second column is then placed to the right of the matrix. Player A examines this column and chooses first row corresponding to the *largest* number in this column. This row is now added to the row last chosen and the sum of the two rows is placed below the row last chosen. Player B then chooses a column corresponding to the smallest number in the new row and adds this column to the column last chosen. The procedure is repeated for a finite number of iterations. Three iterations are shown below with the smallest elements in each succeeding row and the largest elements in each succeeding column encircled:

	Player B						
	0	1	–1	(1)	0	0	1/3
Player A	–1	0	1	0	(1)	0	1/3
	1	–1	0	–1	–1	(0)	1/3
	1	(–1)	0				
	1	0	(–1)				
	(0)	0	0				
	1/3	1/3	1/3				

The approximate strategies after 3 iterations (further iterations are not possible as all the three elements in the succeeding rows and columns turn out to be zero) are found by dividing the number of encircled elements in each row or column by the total number of iterations. Thus, optimal mixed strategies for players A and B are

Player A: A_1 : 1/3, A_2 : 1/3, A_3 : 1/3, Player B: B_1 : 1/3, B_2 : 1/3, B_3 : 1/3

The value of the game $V = 0$.

CC203 OR: Operations Research

Module – II

Unit 5: The Replacement Problem

(Dr. Natasa Dasgupta)

The replacement problems deal with the situation that arise when some components (or men or machinery) requires replacement because of **reduced efficiency**, or **breakdown** or **complete failure.** Such decreased efficiency or complete failure may be either gradual or all of a sudden.

The need for decision of replacement is raised in any organization both in case of men and machinery.

Objectives of Replacement

The primary objective of replacement is to direct the organization towards **profit maximization or cost minimization**. Deciding the replacement policy that determines the optimal replacement age of equipment, instead of using with higher maintenance costs for long time, is the main objective of replacement problem.

Types of Replacement Situations

The replacement situations may be classified into four categories:

a) Replacement of items that become worse with time e.g. automobile tyres, milk plant machinery, tools, vehicles, equipment etc.

b) Replacement of items which do not deteriorate with time but break down completely after certain usage e.g. electric tubes, machinery parts etc.

c) Replacement of items that becomes obsolete due to new developments e.g. mobile phones, software e.t.c.

d) The existing working staff in an organization gradually reduces due to death, retirement and other reasons.

The problem is to decide the best policy to adopt with regard to replacement.

Need for replacement arises in a number of different situations so that different types of decisions may have to be taken.

For example:

a) It may be necessary to decide whether to wait for certain items to fail, which might cause some loss, or to replace the same in advance, even at a higher cost.

b) An item can be considered individually to decide whether or not to replace immediately.

c) It is necessary to decide whether to replace by the same item or by an improved type of item.

Failure Mechanism of Items

Failures can be discussed under two categories viz., Gradual Failures, and Sudden Failures.

A) Gradual Failure

As the life of an item increases, its efficiency deteriorates, causing:

- Increased expenditure for operating costs
- Decreased equipments‟ productivity”
- Decrease in the value (resale or scrap value) of the equipment

Example: bearings, pistons, piston rings, “Automobile Tyres‟, mechanical systems like machines, machine tools, flexible manufacturing equipment etc. fall under this category.

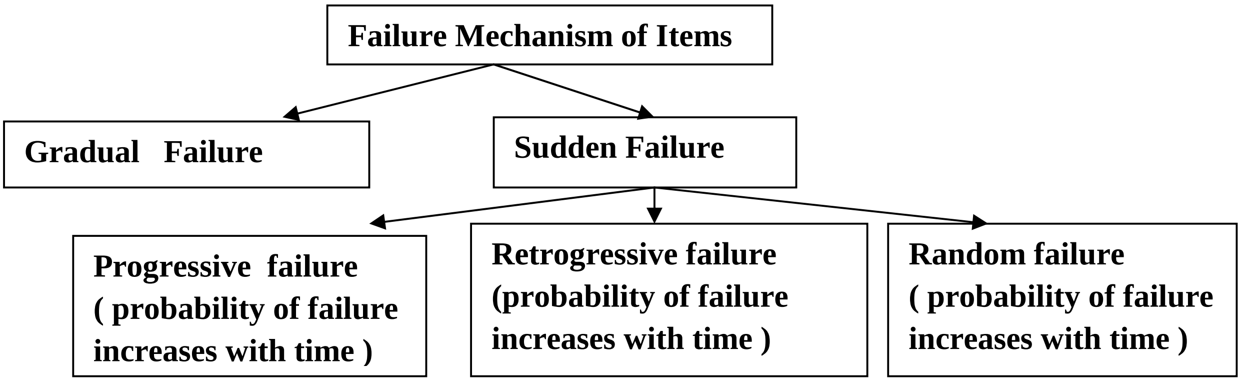
B) Sudden Failure

This type of failure is applicable to those items that do not deteriorate markedly with service, but which ultimately fail after some period of using.

a) Progressive failures: In this mechanism, **probability of failure increases with time** i.e. as the life of equipment increases. Examples include: e*lectric light bulbs, automobile tubes* etc.,

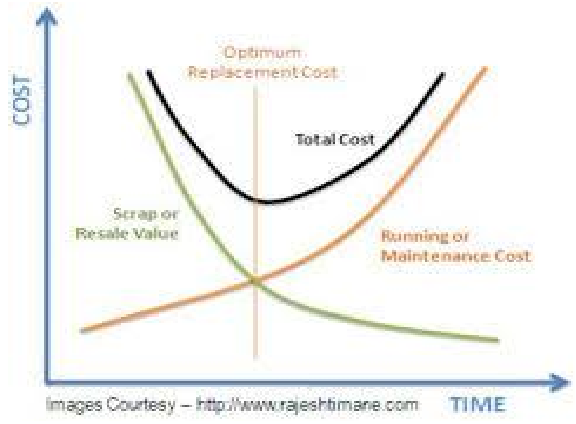
b) Retrogressive failures: Some equipment may prone to failure in the beginning of their life and the **probability of failure falls down with time**. Example: *aircraft engines.*

c) Random failures: Under this failure, **constant probability of failure with time** is associated with the equipment that fails from random causes such as physical shocks. Example: Electronic components like transistors, semi conductor elements, glass made items, delicate or brittle items.



Replacement Policy

When a machine loses its efficiency gradually the maintenance becomes very expensive. Therefore, the problem is to determine the age at which it is most economical to replace the item. On the other hand, certain items such as bulbs, radio, television, and computer parts fail suddenly without giving any indication of failure and they become completely useless. These items are to be replaced immediately as and when they fail to function.



Replacement of Items Deteriorating with Time

The maintenance cost of the items which deteriorate with time always increase gradually with time and a stage comes when the maintenance cost becomes so large that it is better and economical to replace the item with a new one. There may be number of alternatives and we may have a comparison between various alternatives by considering the costs due to waste, scrap, loss of output, damage to equipment and safety risks etc.

Replacement of items(or equipments) whose maintenance cost increases with time and the value of money remains same during the period

The following costs are considered in such decisions:

C = Capital cost of the item,

R(t) = Operating and Maintenance cost of the item at time t,

S = Scrap value of the item,

n = number of years the item is to be in use,

T(n) = Total Cost incurred during n years

Tavg = Average annual cost of the item $= \frac{T(n)}{n}$

Obviously, annual cost of the item at any time t = C –S + R(t)

The operating cost function R(t) is assumed to be strictly positive. It may be continuous or discrete.

Working Formula: When t is a continuous variable, Replace item in the nth year if

$$R(n) = \frac{T(n)}{n}$$

$$\text{where} \quad T(n) = C - S + \int_{0}^{n} R(t)dt$$

Result-1:"If time is measured continuously, then the average annual cost will be minimized by replacing the machine when the average cost today becomes equal to the current maintenance cost".

Working Formula :When t is a discrete variable, Replace item at the end of the nth year if

$$R(n) < \frac{T(n)}{n} < R(n+1)$$

$$\text{where} \quad T(n) = C - S + \sum_{0}^{n} R(t)$$

Result-2: "If time is measured in discrete units, then the average annual cost will be minimized by replacing the machine when the next period's maintenance cost becomes greater than the current average cost".

Example-1: A firm is considering replacement of a machine whose cost price is Rs. 12200 and the scrap value is Rs. 200. The maintenance costs R(t) are found from experience to be as follows;

Year	1	2	3	4	5	6	7	8
Maintenance Cost (Rs)	200	500	800	1200	1800	2500	3200	4000

When should the machine be replaced?

Solution The computations can be summarized in the following tabular form:

Table 1: Calculations for average cost of machine

Replace at the end of the year (n) [1]	Maintenance cost[R(t)] [2]	Total maintenance cost $\sum_0^n R(t)$ [3]	Difference between Price & scrap value (C-S) [4]	Total cost T(n) [5] =[3]+[4]	Average Cost T_{avg} = T(n)/n [6]=[5]/[1]
1	200	200	12000	12200	12200
2	500	700	12000	12700	6350
3	800	1500	12000	13500	4500
4	1200	2700	12000	14700	3675
5	1800	4500	12000	16500	3300
6	2500	7000	12000	19000	3166.6
7	3200	10200	12000	22200	3171.4
8	4000	14200	12000	26200	3265

Since R(6) < T(6)/6 < R(7), the machine should be replaced at the end of the 6^{th} year.

Example 2

A Machine owner finds from his past records that the maintenance costs per year of a machine whose purchase price is Rs. 8000 are as given below:

Year:	1	2	3	4	5	6	7	8
Maintenance Cost:	1000	1300	1700	2200	2900	3800	4800	6000
Resale Price:	4000	2000	1200	600	500	400	400	400

Determine at which time it is profitable to replace the machine.

Solution C = Rs. 8000. Table 2 shows average cost per year during the life of machine.

Table 2: Calculations for average cost of machine

Replace at the end of the year (n) [1]	Maintenance cost[R(t)] [2]	Total maintenance cost $\sum_0^n R(t)$ [3]	Resale value (S) [4]	Total cost T(n) [5] = [3]+ 8000-[4]	Average Cost $T_{avg} = T(n)/n$ [6]=[5]/[1]
1	1000	1000	4000	5000	5000
2	1300	2300	2000	8300	4150
3	1700	4000	1200	10800	3600
4	2200	6200	600	13600	3400
5	2900	9100	500	16600	3200
6	3800	12900	400	20500	3417
7	4800	17700	400	25300	3614
8	6000	23700	400	31300	3913

The above table shows that the value of T_{avg} during fifth year is minimum. Hence the machine should be replaced after every fifth year.

Try yourself (Answer : at the end of the 6th year)

The cost of a machine is Rs. 6100 and its scrap value is only Rs.100. The maintenance costs are found to be

Year:	1	2	3	4	5	6	7	8
Maintenance Cost (in Rs.):	100	250	400	600	900	1250	1600	2000

When should the Machine be replaced?

Replacement of items whose maintenance cost increases with time and the money value changes at a constant rate

To understand this let us define the following terms:

Time Value of Money:

Conceptually "time value of money" means that the value of a unit of money is different in different time periods. If the discount rate is 10%, Rs.100 today is equal to Rs.110 a year from now. Consequently Rs. 1 from a year now is equal to $(1+0.1)^{-1}$ rupee today. The present worth of a rupee received after some time will be less than a rupee received today.

Rational investors would prefer current receipt to future receipts.

Present Worth Factor (PWF):

The value of money over a period of time , depends upon the nominal interest rate "r". The present value of a rupee to be spent after n years is equal to $(1+r)^{-n}$, and is called as **Present Worth Factor (PWF) or Present Value** of one rupee spent n years from now. If r=10%, then PWF in 5 years from now is $(1+10/100)^{-5} = (1.1)^{-5}$.

The term $v = \frac{100}{100+r}$ is called **Discount Rate.** In the above example $v = \frac{100}{110} = (1.1)^{-1}$

Thus, the discounted cost of Rs 100 after n^{th} year = $100\ v^n$

Example 3: The value of money is assumed to be 10% per year and the cost pattern for two machines A and B is given below:

Year	1	2	3
Machine A	900	600	700
Machine B	1400	100	700

Determine which machine is less costly.

Solution: Here, $r = 100,\ v = \frac{100}{110} = \frac{10}{11}$

Table 3: Calculation of Present worth of the machine

Year [1]	Yearly cost of A [2]	Discounted Yearly cost of A [3]=[2] * v^{n-1}	Yearly cost of B [4]	Discounted Yearly cost of B [5]=[4] * v^{n-1}
1	900	900	1400	1400
2	600	$600 \times \frac{10}{11}$ = 545.45	100	$100 \times \frac{10}{11}$ = 90.9
3	700	$700 \times (\frac{10}{11})^2$ = 578.52	700	$700 \times (\frac{10}{11})^2$ = 578.52
TOTAL	2200	2023.97	2200	2069.43

Hence we observe that though the total cost of the machines are the same (Rs. 2200 0nly) machine A is cheaper than Machine B considering the money value.

Replacement of Items whose Maintenance and Repair Cost Increases With Time, Value of Money also Changes With Time

The optimal replacement policy when the maintenance cost increases with time, and the money value decreases in a constant discount rate can be determined as follows;

Suppose that the purchasing cost of an item (which may be machine or equipment) is C. R(n) be the operating cost in the n^{th} year. If the item operated for n years, discounted cost associated with the item is

$$F(n) = [C + R(1)] + vR(2) + v^2R(3) + \cdots..+v^{n-1}R(n)$$

Let the item be replaced at the end of every n^{th} year. Then the operating cost form n+1 to 2n in present money value is $(n)v^n$, 2n+1 to 3n is $F(n)v^{2n}$ and so on.

Thus the present worth of all future discounted costs or ***Weighted Average Cost*** (T(n)) associated with the policy of replacement is given by

$$T(n) = F(n) + F(n)v^n + F(n)v^{2n} + \cdots$$

$$= \frac{F(n)}{1-v^n}$$

Working Formula 3: Replace at the end of the n^{th} year when T(n) is minimum:

$$R(n) < \frac{C + R(1) + vR(2) + v^2R(3) + \cdots + v^{n-1}R(n)}{1 + v + v^2 + \cdots + v^{n-1}} < R(n+1)$$

i.e

$$R(n) < T(n) < R(n+1)$$

Result 3: *Replace if the next period's maintenance cost is greater than the weighted average cost of previous periods. Do not replace if the next period's cost is less than the weighted average of the previous costs."*

Example 4

A milk plant is offered an equipment A which is priced at Rs.60,000 and the costs of operation and maintenance are estimated to be Rs.10,000 for each of the first 5 years, increasing every year by Rs. 3000 per year in the sixth and subsequent years. If money carries the rate of interest 10% per annum what would the optimal replacement period?

Solution

Table 4 Determination of optimal replacement period

At the end of year (n)	Operating & maintenance cost R_n	Discounted factor v^{n-1}	Discounted operation & maintenance cost $R_n v^{n-1}$	Cumulative Discounted operation & maintenance cost	Discounted total cost $T(n) = C + \sum v^{n-1} R(n)$	Cumulative discounted factor $\sum v^{n-1}$	Weighted average annual cost $F(n) = \frac{T(n)}{\sum v^{n-1}}$
(1)	(2)	(3)	(4)=(2)x(3)	(5)	(6)=(5)+60000	(7)	(8)=(6)/(7)
1	10000	1.0000	10000.00	10000.00	70000.00	1.00	70000.00
2	10000	0.9091	9091.00	19091.00	79091.00	1.91	41428.42
3	10000	0.8264	8264.00	27355.00	87355.00	2.74	31933.83
4	10000	0.7513	7513.00	34868.00	94868.00	3.49	27207.75
5	10000	0.6830	6830.00	41698.00	101698.00	4.17	24389.18
6	13000	0.6209	8071.70	49769.70	109769.70	4.79	22913.08
7	16000	0.5645	9032.00	58801.70	118801.70	5.36	22184.36
8	19000	0.5132	9750.80	68552.50	128552.50	5.87	21905.89
9	22000	0.4665	10263.00	78815.50	138815.50	6.33	21912.82
10	25000	0.4241	10602.50	89418.00	149418.00	6.76	22106.52

From Table 13.4 we find the weighted cost is minimum at the end of 8th year, [19000< 21905.89 <22000] hence the equipment should be replaced at the end of 8th year.

Example 5:

A Manufacturer is offered two machines A and B. Machine A is priced at Rs. 5000 and running cost is estimated at Rs. 800 for each of the first five years, increasing by Rs. 200 per year in the sixth and subsequent years. Machine B, with the same capacity as A, costs Rs. 2500, but has running cost of Rs. 1200 per year for six years, thereafter increasing by Rs. 200 per year. If money is worth 10% per year, which machine should be purchased? (Assume that the machines will eventually be sold for scrap at a negligible price).

Solution

Since money is worth 10% per year, therefore discount rate is $v = \frac{1}{(1+0.10)} = 0.9091$

Table 5a: Computation of weighted average cost for machine A

At the end of year (n)	Operating & maintenance cost R_n	Discounted factor v^{n-1}	Discounted operation & maintenance cost $R_n v^{n-1}$	Cumulative Discounted operation & maintenance cost	Discounted total cost $C + \sum R^{n-1}$	Cumulative discounted factor $\sum v^{n-1}$	Weighted average annual cost $\frac{C + R_n v^{n-1}}{\sum v^{n-1}}$
(1)	(2)	(3)	(4)=(2)x(3)	(5)	(6)=(5)+ 5000	(7)	(8)=(6)+(7)
1	800	1.0000	800	800	5800	1	5800
2	800	0.9091	727	1527	6527	1.9091	3419.035
3	800	0.8264	661	2188	7188	2.7355	2627.819
4	800	0.7513	601	2789	7789	3.4868	2233.98
5	800	0.6830	546	3336	8336	4.1698	1999.098
6	1000	0.6209	621	3957	8957	4.7907	1869.61
7	1200	0.5645	677	4634	9634	5.3552	1799.025
8	1400	0.5132	718	5353	10353	5.8684	1764.13
9	1600	0.4665	746	6099	11099	6.3349	1752.043
10	1800	0.4241	763	6862	11862	6.759	1755.053

From table 5a Since the running cost of 9th year is 1600 and that of 10th year is 1800 and 1600<1752.043<1800, it would be economical to replace machine A at the end of nine years.

Table 5b Computation of weighted average cost for machine B

At the end of year (n)	Operating & maintenance cost R_n	Discounted factor v^{n-1}	Discounted operation & maintenance cost $R_n v^{n-1}$	Cumulative Discounted operation & maintenance cost	Discounted total cost $C+\sum R^{n-1}$	Cumulative discounted factor $\sum v^{n-1}$	Weighted average annual cost $\frac{C+R_n v^{n-1}}{\sum v^{n-1}}$
(1)	(2)	(3)	(4)=(2)x(3)	(5)	(6)=(5)+ 2500	(7)	(8)=(6)+(7)
1	1200	1.0000	1200.00	1200.00	3700.00	1.00	3700.00
2	1200	0.9091	1090.92	2290.92	4790.92	1.91	2509.52
3	1200	0.8264	991.68	3282.60	5782.60	2.74	2113.91
4	1200	0.7513	901.56	4184.16	6684.16	3.49	1916.99
5	1200	0.6830	819.60	5003.76	7503.76	4.17	1799.55
6	1200	0.6209	745.08	5748.84	8248.84	4.79	1721.84
7	1400	0.5645	790.30	6539.14	9039.14	5.36	1687.92
8	1600	0.5132	821.12	7360.26	9860.26	5.87	1680.23
9	1800	0.4665	839.70	8199.96	10699.96	6.33	1689.05
10	2000	0.4241	848.20	9048.16	11548.16	6.76	1708.56

In table 5b we find that 1800<1689<2300 so it is better to replace the machine B after 8th year. The equivalent yearly average discounted value of future costs is Rs. 1748.60 for machine A and it is 1680.23for machine B. Hence, it is more economical to buy machine B rather than machine A.

REPLACEMENT OF ITEMS THAT FAIL COMPLETELY AND SUDDENLY

A system generally consists of a huge number of low-priced components that are increasingly liable to failure with age. The costs of failure, in such a case will be fairly more than the cost of the item itself. For example, a tube or a condenser in an aircraft costs little, but the failure of such a low cost item may lead the airplane to crash. Hence we use some replacement policy for such items which would minimize the possibility of complete breakdown. The following are the replacement policies, which are applicable for this situation.

(i) ***Individual replacement policy*** in which an item is replaced immediately after it fails.

(ii) ***Group replacement policy*** in which a decision is made as regard to at what equal intervals, all the items are to be replaced simultaneously irrespective of whether they have failed or not, with a provision to replace the items individually, which fail during the fixed group replacement period.

The optimal period of replacement is determined by calculating the minimum total cost considering the average cost of group replacement and the cost of individual replacement.

Average Cost of group replacement: Here we propose to replace all items at fixed interval t, whether they have failed or not in addition to replacing the failed item when they stop working.

Let N be the total number of unit in the system and N_t be the number of the items failed and hence replaced at the end of the period t.

$C_1 \, and \, C_2$ are the per unit cost of individual replacement and group replacement respectively,

Then C(t), total cost of group replacement after time period t

$$C(t) = C_1[\,N_1 + \cdots .. + N_{t-1}\,] + C_2\,N,$$

Average cost of group replacement after time period t = $C(t)/t$

Working formula 4: Replace the whole lot at the end of the n^th year if

$$C_1\,N^{t-1} < \frac{C(t)}{t} < C_1\,N^{t},$$

Result-4: Group replacement should be made at the end of t^{th} period if the cost of individual replacements for the period t is greater than the average cost per period through the end of the period .

Average Cost of individual replacement:

Let the expected life of the item is M years (unit of time-years/months/weeks/days/hours e.t.c)

- ⇨ Average number of failures in a year 1/M.

⇨ Average number of failure out of a total of N items in the system in that year is N/M

Thus , cost of individual replacement is $C_2\ N/M$

Example 6: The following failure rates have been observed for a certain type of transistors in a digital computer.

End of week	1	2	3	4	5	6	7	8
Probability of failure to death	0.05	0.13	o.25	0.43	0.68	0.88	0.96	1.00

The cost of replacing an individual failed transistor is Rs. 1.25. The decision is to replace all these transistor simultaneously at fixed intervals and to replace the individual transistor as they fail in service. If the cost of group replacement is 30 paise per transistor, what is the best interval between group replacements? Calculate the cost of individual replacement. Which policy would you prefer and why?

Solution: Let p(i) be the probability that a transistor fails during the i^{th} week of life.

Table -6a : Calculation of p(i) and Expectation(Mean) M

End of week (i)	1	2	3	4	5	6	7	8	Mean M
P[failure to death]= Fi	0.05 F1	0.13 F2	0.25 F3	0.43 F4	0.68 F5	0.88 F6	0.96 F7	1.00 F8	
p(i) = F(i)-F(i-1)	0.05 F1	0.08 F2-F1	0.12 F3-F2	0.18 F3-F2	0.25 F3-F2	0.2 F3-F2	0.08 F3-F2	0.04 F3-F2	
ip(i)	0.05	0.16	0.36	0.72	1.25	1.2	0.56	.32	4.62

Calculation of N_t

N_0 = No. of the Transistor at the beginning = 1000

$N_1 = N_0\ p(1)$ = 1000 X .05 = 50

$N_2 = N_0\ p(2) + N_1\ p(1)$ = 1000X0.08+ 50 X 0.05 = 82

$N_3 = N_0\, p(3) + N_1\, p(2) + N_2\, p(1) = 128$

$N_4 = N_0\, p(4) + N_1\, p(3) + N_2\, p(2) + N_3\, p(1) = 199$

$N_5 = N_0\, p(5) + N_1\, p(4) + N_2\, p(3) + N_3\, p(2) + N_4\, p(1) = 289$

$N_6 = N_0\, p(6) + N_1\, p(5) + N_2\, p(4) + N_3\, p(3) + N_4\, p(2) + N_5\, p(1) = 272$

$N_7 = N_0\, p(7) + N_1\, p(6) + N_2\, p(5) + N_3\, p(4) + N_4\, p(5) + N_5\, p(6) + N_6\, p(7) =194$

$N_8 = N_0\, p(8) + N_1\, p(7) + N_2\, p(6) + N_3\, p(5) + N_4\, p(4) + N_5\, p(3) + N_6\, p(2) + N_7\, p(1) = 196$

Table 6b: Calculation of Average cost of replacement

End of week (t)	Individual Replacement(N_t)	Total Cost C(t)	Average cost C(t)/t
1	50	50X1.25+1000X0.3= 363	363
2	132	132X1.25+1000X0.3 = 465	232.50
3	260	260X1.25+1000X0.3 =625	208.3
4	450	450X1.25+1000X0.3=874	18.52

Since average cost is lowest against week 3, the optimum interval between group replacements is 3 weeks.

From first table; Mean of the item = 4.62 weeks

Average cost of the individual replacement = 1000X1.25/4.62 = Rs. 270/week

Since average cost of group replacement is less, the policy of group replacement is better.

Ref:

1. **S.D. Sharma**
2. **Kanti Swarup, PK Gupta, Manmohan**

University of Calcutta

M.Com – Semester II

CC-203 --- Operations Research- Module –II

Dr. Arindam Kundu
4/3/2020

CPM / PERT

One of the most challenging jobs that any manager can take on is the **management of a large-scale project that requires coordinating numerous activities** throughout the organization.

A myriad of details must be considered in planning how to coordinate all these activities, in developing a realistic schedule, and then in monitoring the progress of the project.

PERT (Program Evaluation and Review Technique) **and CPM** (Critical Path Method) are basically time-oriented methods in the sense that they both lead to **determination of a time schedule** for the project.
The significant difference between two approaches is that the **time estimates** for the different activities in **CPM** were assumed to be **deterministic** while in **PERT** these are described **probabilistically**.
These techniques are referred as **project scheduling** techniques.

USED IN: Production management - for the jobs of repetitive in nature where the **activity time estimates can be predicted** with considerable **certainty** due to the existence of **past experience**.

USED IN: Project management - for **non-repetitive jobs (research and development work**), where the time and cost estimates tend to be **quite uncertain**. This technique uses **probabilistic time estimates**.

Applications of CPM / PERT

These methods have been applied to a wide variety of problems in industries and have found acceptance even in government organizations. These include

- Construction of a dam or a canal system in a region
- Construction of a building or highway
- Cost control of a project using PERT / COST

- Designing a prototype of a machine

The Framework for PERT and CPM

Essentially, there are six steps which are common to both the techniques. The procedure is listed below:

I. Define the Project and all of its significant activities or tasks. The Project (made up of several tasks) should have only a single start activity and a single finish activity.

II. Develop the relationships among the activities. Decide which activities must precede and which must follow others.

III. Draw the "Network" connecting all the activities. Each Activity should have unique event numbers. Dummy arrows are used where required to avoid giving the same numbering to two activities.

IV. Assign time and/or cost estimates to each activity

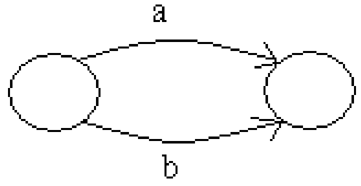
V. Compute the **longest time path** through the network. This is called the **critical path**.

VI. Use the Network to help **plan, schedule, and monitor and control the project**.

ACTIVITY

Any **individual operation** which utilizes resources and has an **end** and a **beginning** is called **activity**. An **arrow** is commonly used to represent an activity with its **head indicating the direction of progress** in the project.

Each activity is represented **by one and only one arrow** in the network.



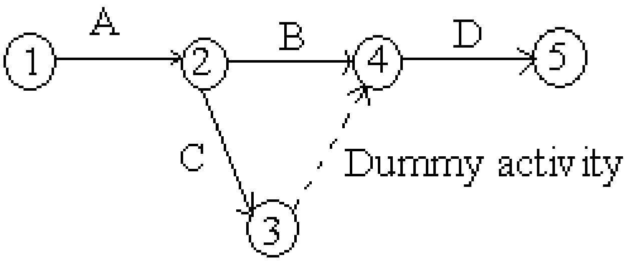
These are classified into four categories -

1. **Predecessor activity** – Activities that must be completed immediately prior to the start of another activity are called predecessor activities.
2. **Successor activity** – Activities that cannot be started until one or more of other activities are completed but immediately succeed them are called successor activities.
3. **Concurrent activity** – Activities which can be accomplished concurrently are known as concurrent activities. It may be noted that an activity can be a predecessor or a successor to an event or it may be concurrent with one or more of other activities.
4. **Dummy activity** – An activity which does not consume any kind of resource but merely depicts the technological dependence is called a dummy activity.

The **dummy activity** is inserted in the network **to clarify the activity pattern** in the following two situations

- To make activities with **common starting and finishing points distinguishable**
- To identify and maintain the **proper precedence relationship between activities that is not connected by events**.

For example, consider a situation where A and B are concurrent activities. C is dependent on A and D is dependent on A and B both. Such a situation can
be handled by using a dummy activity as shown in the figure.



EVENT

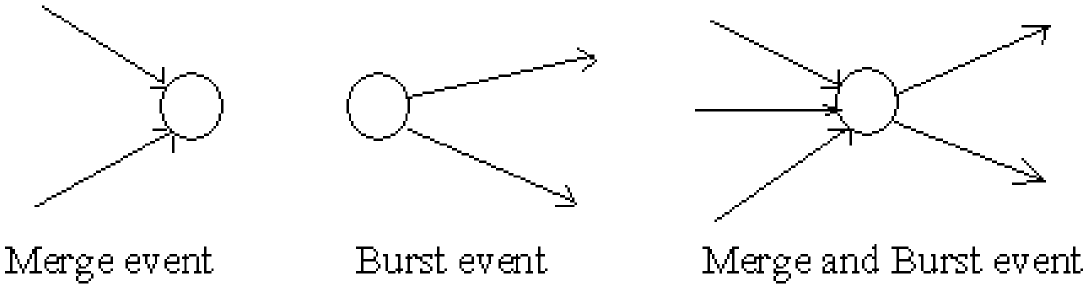
An event represents a point in time signifying the completion of some activities and the beginning of new ones. This is usually represented by a circle in a network which is also called a node or connector.

The events are classified in to three categories:

Merge event – When more than one activity **comes and joins an event** such an event is known as merge event.

Burst event – When more than one activity **leaves an event** such an event is known as burst event.

Merge and Burst event – An activity may be merge and burst event at the same time as with respect to some activities it can be a merge event and with respect to some other activities it may be a burst event.



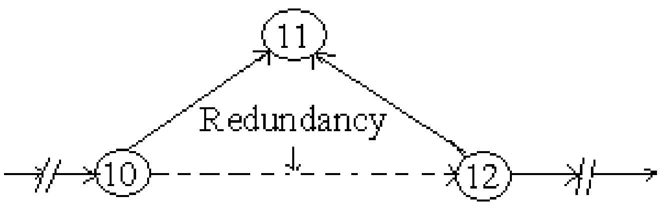
SEQUENCING

The first prerequisite in the development of network is **to maintain the precedence relationships.** In order to make a network, the following points should be taken into considerations

- What job or jobs precede it?
- What job or jobs could run concurrently?
- What job or jobs follow it?
- What controls the start and finish of a job?

REDUNDANCY

Unnecessarily inserting the dummy activity in network logic is known as the error of redundancy as shown in the following diagram



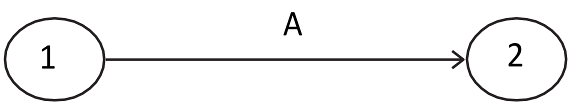
Project Networks

A network used to represent a project is called a **Project Network.** A project network consists of a number of *nodes* (typically shown as small circles or rectangles) and a number of *arcs* (shown as arrows) that lead from some node to another.

Develop a network diagram for the project specified below:

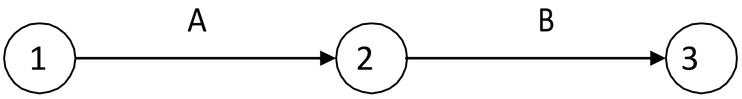
Activity	**Immediate Predecessor Activity**
A	-
B	A
C, D	B
E	C
F	D
G	E, F

Activity A has no predecessor activity. It is the first activity. Let us suppose that activity A takes the project from event 1 to event 2.

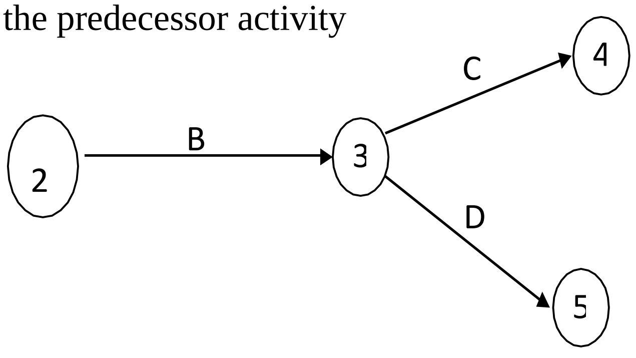


For activity B, the predecessor activity is A.

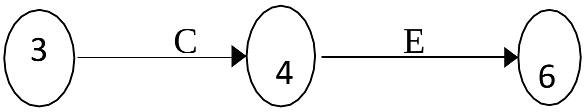
Let us suppose that B joins nodes 2 and 3.



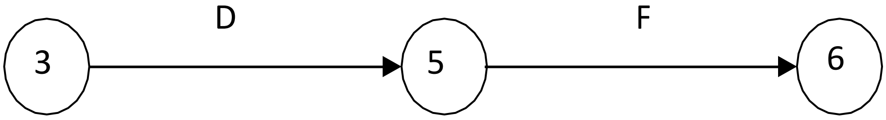
Activities C and D have B as the predecessor activity



Activity E has C as the predecessor activity

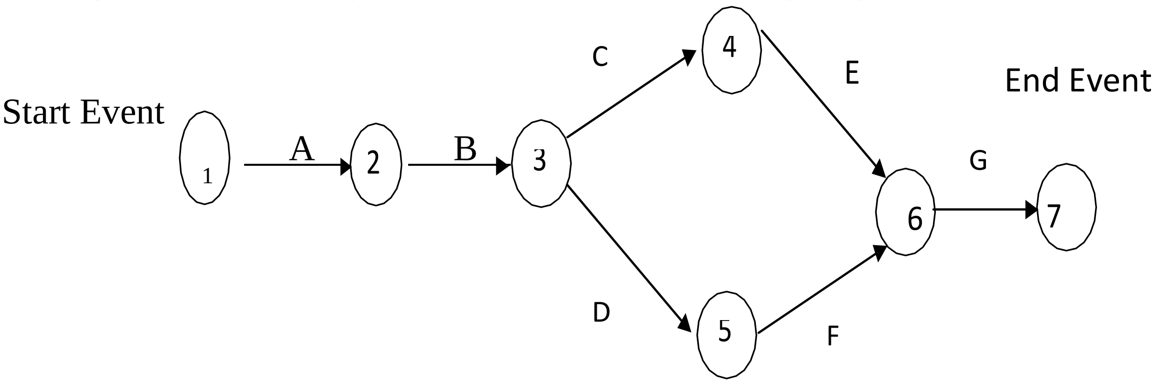


Activity F has D as the predecessor activity



Activity G has E and F as predecessor activities. This is possible only if end nodes E and F are same.

Putting all the pieces together, we obtain the following diagram project network:



The critical path method (CPM) aims at the **determination of the time to complete a project** and the important activities on which a manager shall focus attention.

Project Completion Time

From the start event to the end event, the time required to complete all the activities of the project in the specified sequence is known as the project completion time.

Path in a Project

A continuous sequence, consisting of nodes and activities alternatively, beginning with the start event and stopping at the end event of a network is called a path in the network.

Critical Path and Critical Activities

Consider all the paths in a project, beginning with the start event and stopping at the end event. For each path, calculate the time of execution, by adding the time for the individual activities in that path.

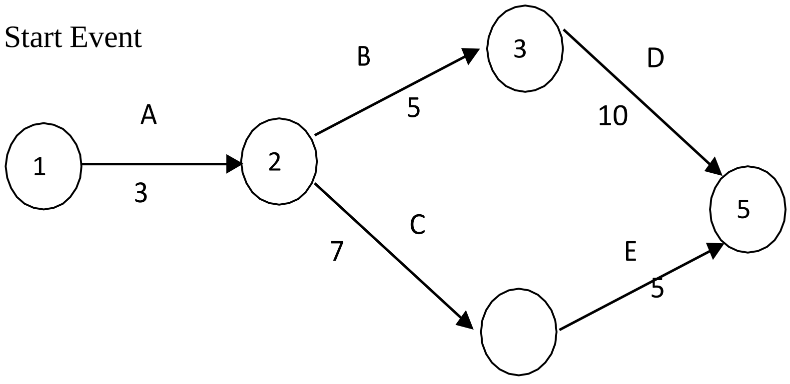
The **path with the largest time is called the critical path** and the **activities along this path are called the critical activities** or bottleneck activities. The activities are called critical because they cannot be delayed. However, a non-critical activity may be delayed to a certain extent.

Any delay in a critical activity will delay the completion of the whole project. However, a certain permissible delay in a non-critical activity will not delay the completion of the whole project. It shall be noted that delay in a non-critical activity beyond a limit would certainly delay the completion the whole project.

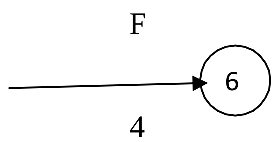
Sometimes, there may be several critical paths for a project. A project manager shall pay special attention to critical activities.

Activity	Predecessor Activity	Duration (Weeks)
A	-	3
B	A	5
C	A	7
D	B	10
E	C	5
F	D,E	4

Determine the critical path, the critical activities and the project completion time.
Network diagram for the project:

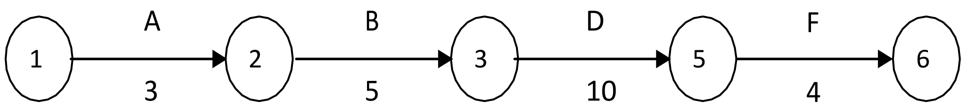


End Event



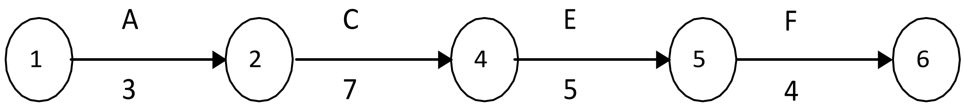
Consider the paths, beginning with the start node and stopping with the end node. There are two such paths for the given project

<u>Path I</u>



Total Time: 3 + 5 + 10 + 4 = 22 weeks.

<u>Path II</u>



Total Time: 3 + 7 + 5 + 4 = 19 weeks

Compare the times for the two paths. Maximum of {22, 19} = 22.

Path I has the maximum time of 22 weeks. Therefore, **Path I is the Critical Path and activities A, B, D and F are Critical Activities**. **Project completion time is 22 weeks.**

Activities C and E (Path II but not in Path I) are Non- Critical activities.

Time for path I - Time for path II = 22- 19 = 3 weeks.

Therefore, together the noncritical activities can be delayed up to a maximum of 3 weeks, without delaying the completion of the whole project.

(i, j) = **Activity** with tail event i and head event j

t_{ij} = **Duration** of activity (i, j)

Earliest occurrence time of event (E_i) – It is the <u>earliest time</u> at which an event can occur without affecting the total project time

Latest occurrence time of event (L_j **) -** It is the latest time at which an event can occur without affecting the total project time

Earliest start time of event – It is the time at which the activity can start without affecting the total project time

Latest start time of event – It is the latest possible time by which an activity must start without affecting the total project time

Earliest finish time of event – It is the earliest possible time at which an activity can finish without affecting the total project time

Latest finish time of event – It is the latest time by which an activity must get completed without delaying total project completion

- **Forward Pass Method - For Earliest Time calculation**
- **Backward Pass method – For Latest Allowable Time calculation**

Floats of an Activity:

Total float – The amount of time by which the **completion of an activity** could be delayed beyond the earliest finish time **without affecting the overall project duration time**

Total float for activity (i – j) = Latest start time for the activity – Earliest start time for the activity

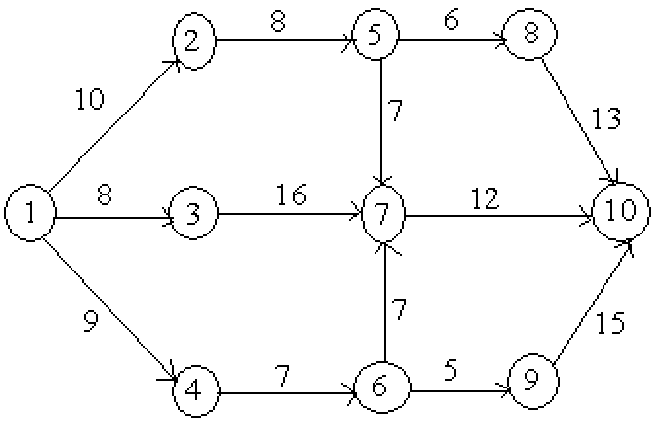
Free float – The time by which the completion of an activity can be delayed beyond the earliest finish time **without affecting the earliest start of a subsequent activity**

Free Float for Activity (i, j) = Earliest occurrence time for Event j – **Earliest** occurrence time for Event i – Duration of Activity (i, j)

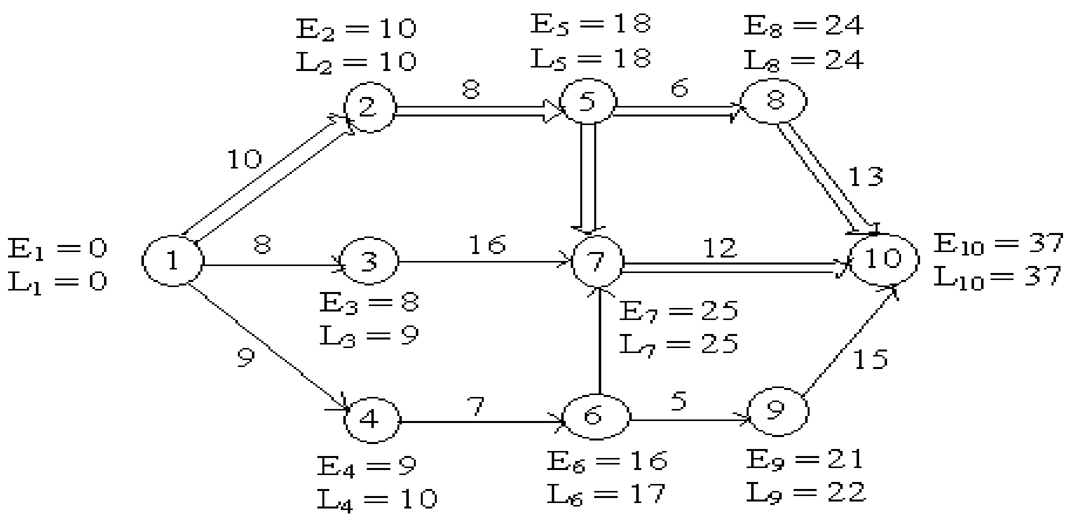
Independent float – The amount of time by which the start of an activity can be delayed **without affecting the earliest start time of any immediately following activities**, assuming that the preceding activity has finished at its latest finish time. The negative independent float is always taken as zero.

Independent Float for Activity (i, j) = Earliest occurrence time for Event j – **Latest** occurrence time for Event i – Duration of Activity (i, j)

Determine the early start and late start in respect of all node points and identify critical path for the following network:



Solution:



Activity (i, j)	Duration	Earliest Time		Latest Time		Total Float Time $(L_j - t_{ij}) - E_i$
		Start (E_i)	Finish $(E_i + t_{ij})$	Start $(L_j - t_{ij})$	Finish (L_j)	
(1, 2)	10	0	10	0	10	0
(1, 3)	8	0	8	1	9	1
(1, 4)	9	0	9	1	10	1
(2, 5)	8	10	18	10	18	0
(4, 6)	7	9	16	10	17	1
(3, 7)	16	8	24	9	25	1
(5, 7)	7	18	25	18	25	0
(6, 7)	7	16	23	18	25	2
(5, 8)	6	18	24	18	24	0
(6, 9)	5	16	21	17	22	1
(7, 10)	12	25	37	25	37	0
(8, 10)	13	24	37	24	37	0
(9, 10)	15	21	36	22	37	1

From the above table, there are two possible Critical Paths

Path I: $1 \rightarrow 2 \rightarrow 5 \rightarrow 8 \rightarrow 10$

Path II: $1 \rightarrow 2 \rightarrow 5 \rightarrow 7 \rightarrow 10$

Earliest time

$E_1 = 0$

$E_2 = 0 + 10 = 10$

$E_3 = 0 + 8 = 8$

$E_4 = 0 + 9 = 9$

$E_5 = 10 + 8 = 18$

$E_6 = 9 + 7 = 16$

E_7 = max {18 + 7, 16 + 7} = 25

E_8 = 18 + 6 = 24

E_9 = 16 + 5 = 21

E_{10} = max {24 + 13, 25 + 12, 21+ 15} = 37

Latest time

L_{10} = 37

L_9 = 37 – 15 = 22

L_8 = 37 – 13 = 24

L_7 = 37 – 12 = 25

L_6 = min {25 – 7, 22 -5} = min {18, 17} = 17

L_5 = min {24 – 6, 25-7} = min {18, 18} = 18

L_4 = 17 – 7 = 10

L_3 = 25 - 16 = 9

L_2 = 18 – 8 = 10

L_1 = min {10 - 10, 9 – 9, 10 - 9} = 0

Project Evaluation and Review Technique (PERT)

The main objective in the analysis through PERT is to find out the completion for a particular event within specified date. The **PERT approach takes into account the uncertainties**. The three time values are associated with each activity

Optimistic time (t_0) - It is the **shortest possible time** in which the activity can be finished. It assumes that **everything goes very well**.

Most likely time (t_m) – This is the most realistic time to complete the activity

If a graph is plotted in the time of completion and the frequency of completion in that time period, then most likely time will represent the highest frequency of occurrence.

Pessimistic time (t_p) – It represents the longest time the activity could take.

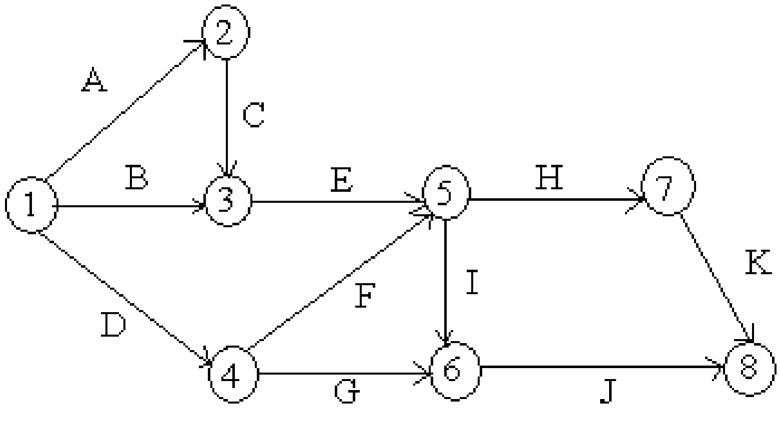
Expected time – It is the average time an activity will take if it were to be repeated on large number of times and is based on the assumption that the activity time follows Beta distribution, this is given by

$$t_e = (t_0 + 4\, t_m + t_p) / 6$$

The **variance** for the activity is given by

$$\sigma^2 = [(t_p – t_o) / 6]^2$$

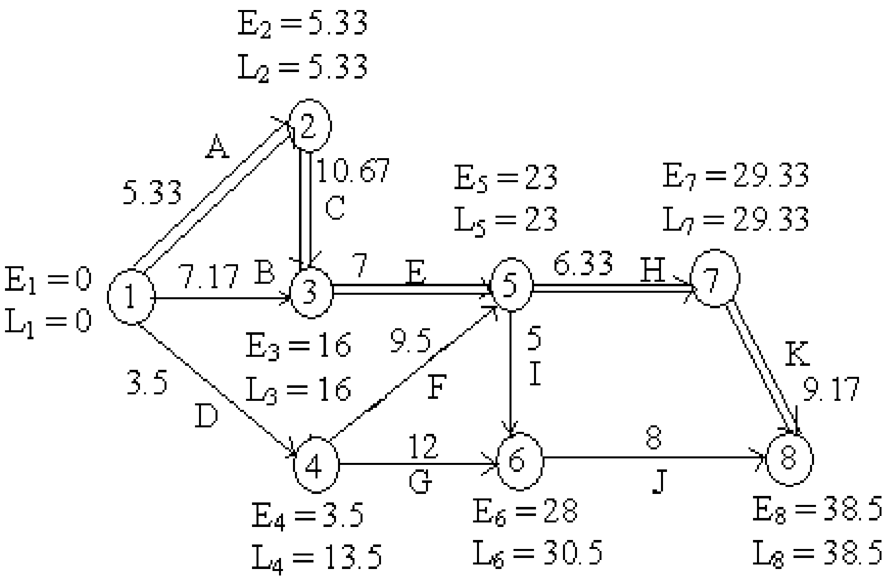
Examples



Task:	A	B	C	D	E	F	G	H	I	J	K
Least time:	4	5	8	2	4	6	8	5	3	5	6
Greatest time:	8	10	12	7	10	15	16	9	7	11	13
Most likely time:	5	7	11	3	7	9	12	6	5	8	9

Find the earliest and latest expected time to each event and also critical path in the network

Task	Least time(t_0)	Greatest time (t_p)	Most likely time (t_m)	Expected time (to + t_p + $4t_m$)/6	Variance
A	4	8	5	5.33	0.44
B	5	10	7	7.17	0.69
C	8	12	11	10.67	0.44
D	2	7	3	3.5	0.69
E	4	10	7	7	1
F	6	15	9	9.5	2.25
G	8	16	12	12	1.78
H	5	9	6	6.33	0.44
I	3	7	5	5	0.44
J	5	11	8	8	1
K	6	13	9	9.17	1.36



The Critical Path is A → C → E → H → K
Expected Project Completion Time: 5.33+10.67+7+6.33+9.17 = 38.5
and Variance 0.44+0.44+1+0.44+1.36 = 3.68

Task	Expected Time (te)	Start Time		Finish Time		Total Float
		Earliest	Latest	Earliest	Latest	
A	5.33	0	0	5.33	5.33	0
B	7.17	0	8.83	7.17	16	8.83
C	10.67	5.33	5.33	16	16	0
D	3.5	0	10	3.5	13.5	10
E	7	16	16	23	23	0
F	9.5	3.5	13.5	13	23	10
G	12	3.5	18.5	15.5	30.5	15
H	6.33	23	23	29.33	29.33	0
I	5	23	25.5	28	30.5	2.5
J	8	28	30.5	36	38.5	2.5
K	9.17	29.33	29.33	31.5	38.5	0

Earliest time

$E_1 = 0$

$E_2 = 0 + 5.33 = 5.33$

$E_3 = \max \{5.33 + 10.67, 0+7.17\} = \max \{16, 7.17\} = 16$

$E_4 = 0 + 3.5 = 3.5$

$E_5 = \max \{16 + 7, 3.5 + 9.5\} = \max \{23, 13\} = 23$

$E_6 = \max [23 + 5, 3.5 + 12] = \max \{28, 15.5\} = 28$

$E_7 = 23 + 6.33 = 29.33$

$E_8 = \max \{29.33 + 9.17, 28 + 8\} = 38.5$

Latest time

L_8 = 38.5

L_7 = 38.5 – 9.17 = 29.33

L_6 = 38.5 – 8 = 30.5

L_5 = min {29.33 – 6.33, 30.5 -5} = min {23, 30} = 23

L_4 = min {23-9.5, 28-12} = min {13.5, 16} = 13.5

L_3 = 23 - 7 = 16

L_2 = 16 – 10.77 = 5.33

L_1 = min {5.33-5.33, 16 – 7.17, 3.5-3.5} = 0

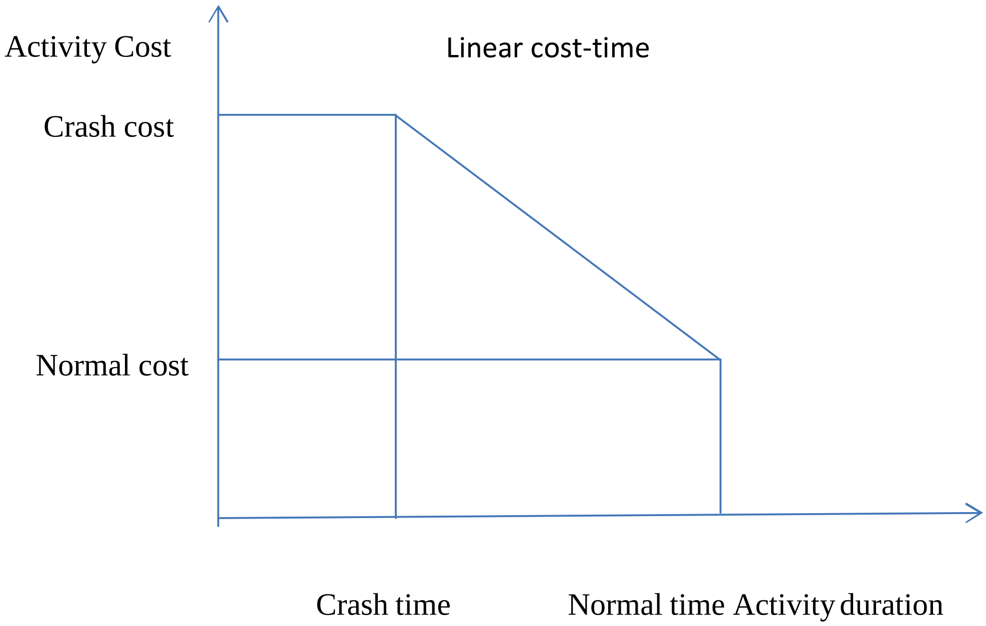
As we are expecting the variability in the activity duration, the total project may not be completed exactly in time. Thus it is necessary to **calculate the probability of actually meeting the scheduled time** to the project as well as activities.

The probability distribution of time for completing an event can be approximated by the normal distribution due to central limit theorem. Thus the probability of completing the project by scheduled time (T_s) is given by
Prob $(Z < (T_s – T_e)/ \sigma)$
Standard normal variate value is given by
$Z = (T_s – T_e)/ \sigma$
T_e = expected completion time of the project

Crashing of a Project

In the project management generally there is a specific date for the project completion. In order to complete the project in less than the normal time, the normal duration of the project must be reduced to the desired duration. The method of reducing the project duration by shortening time of one or more activities at a cost is called crashing.

It is usually achieved by putting into service additional labour or machines to one activity or more activities. Crashing involves more costs. A project manager would like to speed up a project by spending as minimum extra cost as possible. Project crashing seeks to minimize the extra cost for completion of a project before the stipulated time.



Linear time and cost trade-off for an activity

For further practice, please refer

1. Operations Research- Theory and Applications – J.K.Sharma

University of Calcutta

M.Com – Semester II

CC-203 --- Operations Research- Module –II

Dr. Piyali Dutta Chowdhury

4/3/2020

INVENTORY CONTROL MODELS

- Model I – *EOQ model with constant rate of Demand* along with basic theoretical concept were discussed both in the respective Day and Evening sections.

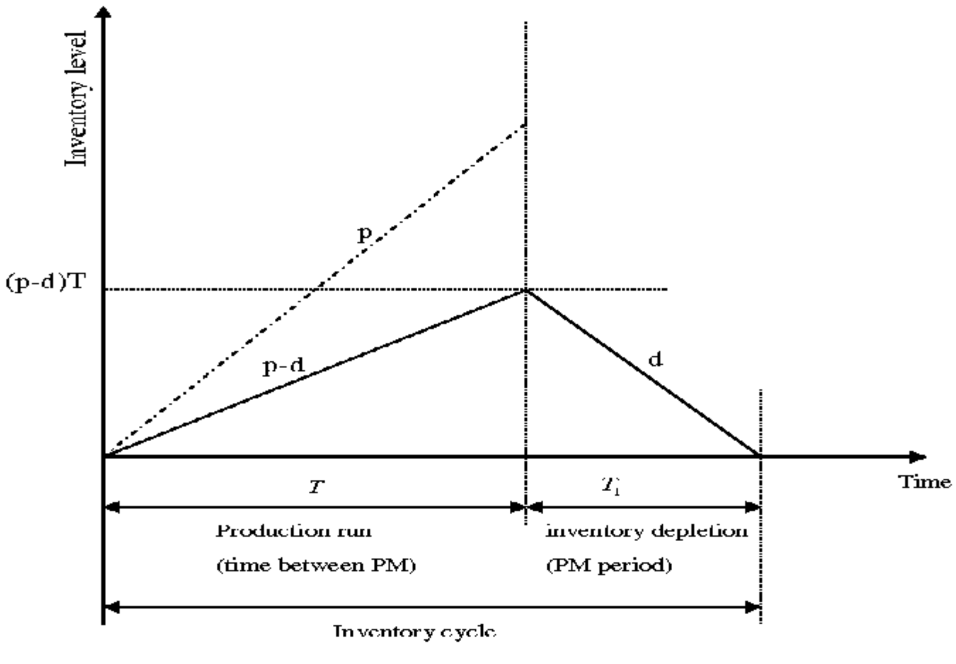
Model II- EOQ model when Supply is Gradual:-

This model is applicable when inventory continuously builds up over a period of time after placing an order or when the units are manufactured and used or sold at a constant rate. This model is specially suitable for the manufacturing environment where there is a simultaneous production and consumption, it is known as “Production Model”.

Assumptions:-

i) The item is sold or consumed at a constant demand rate which is known.
ii) Set up cost is fixed and it does not change with lot size.
iii)The rate of receipts i.e. production rate ‘p’ is greater than the demand or consumption rate‘d’.
iv) Production runs to replenish inventory are made at regular interval ‘t’ and consumption takes place during the entire cycle ‘$T+T_1$’.

Figure- 1 Inventory Model for Non-Instantaneous Replenishment



Notations :-

- Inventory under this situation builds at the rate of (p-d) units and inventory is maximum at the end of production period T.

  I max = (p – d) * T.

  Average Inventory = $\frac{(p-d)*T}{2}$

Now the quantity produced during production period Q = p * T

$$T = \frac{Q}{p},$$

Average Inventory = $\frac{(p-d)}{2} * \frac{Q}{p}$

- Annual Inventory Carrying cost

  $= \frac{Q}{2} * \frac{(p-d)}{p} * C_h$

- Annual Set up cost

  $= \frac{D}{Q} * C_o$

- Therefore, Total Annual Cost = Annual Set up cost + Annual Inventory Carrying cost

$$\textbf{TC} = \frac{D}{Q} * C_o + \frac{Q}{2} * \frac{(p-d)}{p} * C_h$$

To determine the EOQ differentiate the above equation with respect to Q

$$\frac{dTc}{dQ} = -\frac{DC_o}{Q^2} + \frac{1}{2} * \frac{(p-d)}{p} * C_h$$

And equating to '0'and solving we get

$$Q^* = \sqrt{\frac{2DC_o * p}{C_h * (p-d)}}$$

If the inventory carrying cost is expressed as a percentage of annual inventory investment,

$$Q^* = \sqrt{\frac{2DC_o * p}{C_p * I * (p-d)}}$$

- Optimal Number of Production per year

  $N^* = \frac{D}{Q^*}$

- Optimal Production Cycle time

  $t^* = \frac{Q^*}{D}$

- Optimal length of each lot size production run
  $T^* = \frac{Q^*}{p}$

- The optimal total inventory variable cost
  $TC(Q^*) = \sqrt{\frac{2D\, C_o\, C_h (p-d)}{p}}$

Problem 1

The demand rate for a particular item is 48000 units per year, the firm can produce at a rate of 800 units per day.The set up cost is Rs 45/- per item. Carrying cost is Rs 2/- per unit per year. If no shortages are allowed and the replacement is instantaneous determine,

i) the EOQ
ii) optimum annual cost
iii) optimal cycle time
iv) run time
v) maximum inventory level

SOLUTION:- Given,
D = 48000, p = 800 per day C_o = 45/- C_h = 2.00 per year
Daily demand- 48000/240 = 200, assuming in a year there are 240 are working days

→ i) $Q^* = \sqrt{\frac{2D C_o * p}{C_h * (p-d)}}$

$= \sqrt{\frac{2*48000*45*800}{2*(800-200)}}$ = 1697 units.

ii) $TC(Q^*) = \sqrt{\frac{2D\, C_o\, C_h (p-d)}{p}}$

$= \sqrt{\frac{2*48000*45*2*(800-200)}{800}}$

=2545.58/-

iii) $t^* = \frac{Q^*}{D}$

= (1697/48000)*240days = 8.485 days

iv) $T^* = \frac{Q^*}{p}$

= 1697/800 = 2.12 days.

v) I max = $(p-d) * T^*$

= 600*2.12 = 1272 units.

Problem 2

A manufacturing company uses an EOQ approach in planning its production of machinery parts. The following in formations are available as follows-

D = 90,000 parts /day

C_p = 200/- per parts

C_o=4000/- per parts

Inventory carrying cost per month is established at 2% of the average inventory value

Production rate 400 units per day, and the company works for 300 days in a year, the calculate

i) EOQ

ii) the number of production runs per year

iii) production cycle time

iv) total inventory cost.

SOLUTION:- Given, D = 90000, p = 400 per day C_o = 4000/- C_p =200/-

$C_h = C_p$* I= 200*12*0.02=48

Daily demand (d) = 90000/300 = 300, assuming in a year there are 3000 are working days

p> d----

→ i) $Q^* = \sqrt{\frac{2DC_o * p}{C_h * (p-d)}}$

$= \sqrt{\frac{2*90000*4000*400}{48*(400-300)}}$ = 7745.966 units. = 7746 units.

II) $N^* = \frac{D}{Q^*}$ = 90000/7746 = 11.61~ 12 production runs per annum.

iii) $t^* = \frac{Q^*}{D}$ = (7746/90000)*300 = 25.82 ~ 26 days

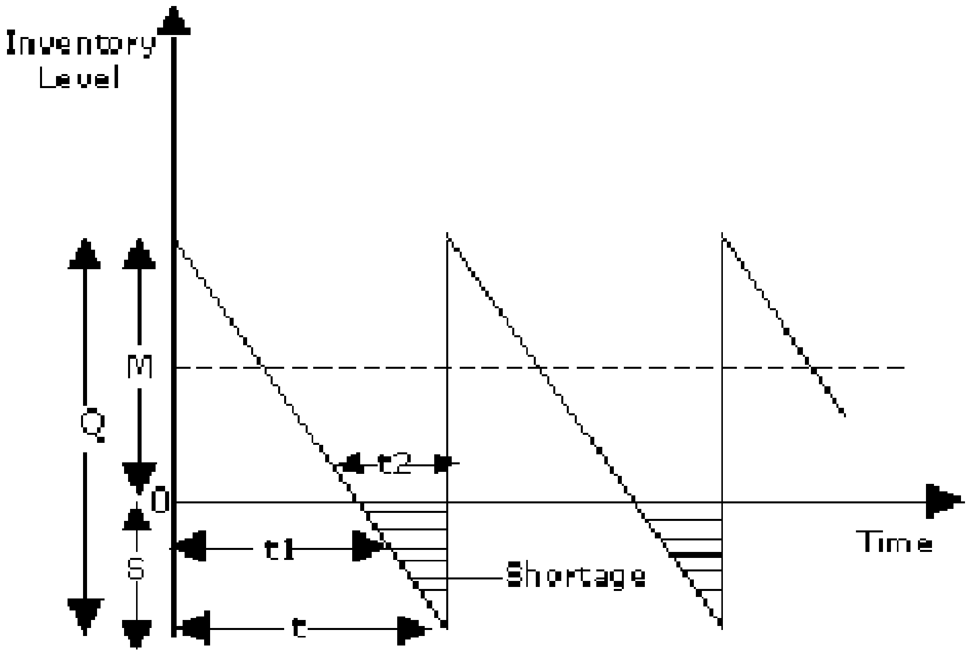
iv) $TC(Q^*) = \sqrt{\frac{2D\, C_o\, C_h(p-d)}{p}}$

$= \sqrt{\frac{2*90000*4000*48*(400-300)}{400}}$ = 92951.60/-

Model III- EOQ model with Shortages:-

Under this model, the inventory system runs out of stock for a certain period of time, i.e. shortages are allowed and the cost of shortage is assumed to be directly proportional to the average number of units short. There are many situations in which planned shortages or stock-outs may be advisable especially for expensive items that have high carrying cost. The model is called the back – order or planned shortages inventory model.

Figure- 2 Change in Inventory over Time with Back-orders



Explanation:-

- Every time the quantity ‘Q’ i.e. order size is received.
- Number of shortages S per order i.e. Back order quantity.

- (Q – S)= M is remaining units after the Back order is satisfied.
- t_1 is the time during which inventory is on hand
- t_2 is the time during which shortage exists
- T is the time between receipts of orders, i.e. $T = t_1 + t_2$

  Assumptions:-

  - The scheduling time period T is constant
  - Production rate is infinite
  - Lead time is zero
  - Sales will not be lost due to stock out

The given figure describes the change in the inventory level with time. Every time the quantity 'Q' is received, all shortages equal to an amount 'S' are first taken care and the remaining quantity (Q – S) = M is placed in inventory as the surplus from which demand during the next cycle will be satisfied. Here it may be noted that 'S' units out of 'Q' are always in the shortage list, i.e. they are never carried in the stock. Thus it yields savings on the inventory carrying cost.

Therefore, in the inventory system except for the purchase cost 'C', which is fixed, all types of costs (assuming 'C_h' as inventory carrying cost and 'C_s' as shortage cost) will be affected by the decision concerning Q^* i.e. the optimal value of order quantity, the optimal stock level $M^* = (Q^* - S^*)$ and optimal shortage level S^*.

The results (without proof) of the above model can be summarized as follows:-

1. Economic Order Quantity

$$Q^* = \sqrt{\frac{2\,D\,C_o\,(C_s + C_h)}{C_h * C_s}}$$

2. Maximum Number of back orders

$S^* = Q^* - M^*$ OR $S^* = Q^* \left(\frac{C_h}{C_h + C_s}\right)$

3. Optimum Stock Level

$$M^* = \sqrt{\frac{2\,D\,C_o\,C_s}{C_h\,(C_s + C_h)}}$$

4. Maximum Inventory level

$I_{max} = Q^* - S^*$

5. Time between orders

$$T^* = \frac{Q^*}{D}$$

6. Number of orders per year = $\frac{D}{Q^*}$

7. Total annual Variable cost

$$= \sqrt{2D\,C_o\,C_h}\sqrt{\left(\frac{C_s}{C_s + C_h}\right)}$$

8. Overall annual cost

$$TC(Q^*) = (C * D) + \sqrt{2D\,C_o\,C_h}\sqrt{\left(\frac{C_s}{C_s + C_h}\right)}$$

Problem 1

Given the following data for an item of uniform demand, instantaneous delivery time and back order facility.

Annual Demand= 800 units , Cost of an item = Rs 40

Ordering Cost = Rs 800, Inventory Carrying cost = 40% of per year of stock value

Back Order cost = Rs10, then find out

i) Minimum cost order quantity

ii) Maximum number of back orders

iii) Maximum inventory level

iv) Time between orders

v) Total Annual variable cost

vi) Overall annual cost

Solution:- Given, D = 800, $C_h = C_p * I = 40 * 0.40$ = Rs 16, C_o = Rs 800, C_s = Rs 10

i) $Q^* = \sqrt{\frac{2\,D\,C_o\,(C_s + C_h)}{C_h * C_s}} = \sqrt{\frac{2*800*800*(10+16)}{16*10}}$ = 456 units.

ii) $S^* = Q^* \left(\frac{C_h}{C_h + C_s}\right) = 456 * \left(\frac{16}{16+10}\right)$ = 281 units

iii) $I_{max} = Q^* - S^*$ = 456 – 281 = 175 units

iv) $T^* = \frac{Q^*}{D}$ = $\frac{456}{800}$ = 0.57years, or 0.57*300 = 171 days (assuming in a year 300 working days)

v) Total Annual variable cost = $\sqrt{2D\,C_o\,C_h}\sqrt{\left(\frac{C_s}{C_s+C_h}\right)}$

$= \sqrt{2 * 800 * 800 * 16 * \left(\frac{10}{10+16}\right)}$ = 2806.58/-

Vi) $TC(Q^*) = (C * D) + \sqrt{2D\,C_o\,C_h}\sqrt{\left(\frac{C_s}{C_s+C_h}\right)}$

$= 40*800 + \sqrt{2 * 800 * 800 * 16 * \left(\frac{10}{10+16}\right)}$

= 34807/-

Problem 2

A manufacturing company uses an EOQ approach regarding to a production dealt in by him. The following information are reflected as
Annual demand: 10,000 units, ordering cost – Rs 20 per order, price – Rs 30 per unit,
Inventory carrying cost: 20% of the value of the inventory per year. The company is considering the possibility of allowing some back order to occur and it has been estimated as 30% of the value of inventory.
i) What should be the optimum number of units of the company?
ii) What quantity of the product should be allowed to be backordered, if any?
iii) What will be the maximum quantity of inventory at the any time of the year?
iv) Would you recommend allowing backordering? If so, what would be the annual cost saving by adopting the policy of back ordering?

Solution:-
Given, D = 10,000, C_p = Rs 30, $C_h = C_p * I$ = 30 *0.20= Rs 6, C_o = Rs 20, C_s = 30 * 0.30= Rs 9

i) $Q^* = \sqrt{\frac{2\,D\,C_o\,(C_s+C_h)}{C_h * C_s}}$ = $\sqrt{\frac{2*10000*20*(9+6)}{6*9}}$ = 333.33 units

ii) $S^* = Q^*\left(\frac{C_h}{C_h+C_s}\right)$ = 333 * $\left(\frac{6}{6+9}\right)$ = 133 units

iii) $I_{max} = Q^* - S^*$ = 200 units

iv) Total Annual variable cost = $\sqrt{2D\ C_o\ C_h}\sqrt{\left(\frac{C_s}{C_s + C_h}\right)}$ = $\sqrt{2 * 10000 * 20 * 6 * \left(\frac{9}{9+6}\right)}$ = Rs 1200

when backorder is not permitted the

$$Q^* = \sqrt{\frac{2\ D\ C_o}{C_h}} = \sqrt{\frac{2 * 10000 * 20}{6}} = 258 \text{ units}$$

And Total Annual variable cost (Q^*) = $\sqrt{2D\ C_o\ C_h}$ = $\sqrt{2 * 10000 * 20 * 6}$

= Rs 1549

Since TC (258units) > TC (333units), the company should accept the proposal for back ordering as this will result in saving of (1549 – 1200) = Rs 349 per year.

Problem 3:- ***EOQ model with shortages but production 'p' is greater than demand 'd' rate.***

The demand for an item in a company is 18000 units per year. The company can produce the items at a rate of 3000 per month. The cost of one set up is Rs 500 and the holding cost of 1 unit per month is 15 paise. The shortage cost of one unit is Rs 20.00 per month. Determine

i) Optimum production batch quantity

ii) Number of shortages

iii) Optimum cycle time

iv) Optimal production time

v) maximum inventory level in the cycle

vi) Total associated cost per year

Solution:-

Here, Annual demand D = 18,000 unit, monthly demand 'd' = 1500, production rate 3000 unit i.e. Production rate 'p' > Demand rate 'd'

additionally, $C_h = 0.15$, $C_o =$ Rs 500, $C_s =$ Rs 20

i) Optimum production batch quantity

$$Q^* = \sqrt{\frac{2 * D * C_o * (C_s + C_h) * p}{C_h * C_s * (p - d)}} = \sqrt{\frac{2 * 1500 * 500 * (20 + 0.15) * 3000}{0.15 * 20 * (3000 - 1500)}} = 4489 \text{ units}$$

ii) Number of shortages

$$S^* = Q^* \left(\frac{C_h}{C_h + C_s}\right)\left(\frac{p-d}{p}\right) = 4489 * \left(\frac{0.15}{0.15+20}\right) * \left(\frac{3000-1500}{3000}\right) = 17 \text{ units.}$$

iii) Optimum cycle time $T^* = \frac{Q^*}{d} = \frac{4489}{1500} = 3$ months

iv) Optimal production time $t_1 = \frac{Q^*}{p} = \frac{4489}{3000} = 1.5$ months

v) Maximum inventory level in the cycle

$$M^* = Q^* * \left(\frac{p-d}{p}\right) - S^* = 4489 * \left(\frac{3000-1500}{3000}\right) - 17 = 2228 \text{ units}$$

vi) Total associated cost per year

$$TC^* = \sqrt{2 * D * C_o * C_h * \left(\frac{p-d}{p}\right) * \left(\frac{C_s}{C_s + C_h}\right)}$$

$$= \sqrt{2 * 1500 * 500 * 20 * \left(\frac{3000-1500}{3000}\right) * \left(\frac{20}{20+0.15}\right)}$$

= Rs 3859

Model IV- EOQ model with Price Discounts:-

When items are bought in large quantities, the supplier often gives discount. However, if the material is purchased to take advantage of discount, the average inventory level and so inventory carrying cost will increase. Benefits for the purchaser from large orders are, lower cost per unit, lower shipping and transportation cost, reduced handling cost and reduction in ordering costs due to less number of orders.

These benefits are to be compared with the increase in carrying cost. As the order size increases, more spaces should be provided to stock the items.

A decision is, therefore, to be taken whether the buyer should stick to economic order quantity or increase the same to take advantage that, at large quantities , the production costs per piece are lower and hence, part of the savings can passed on to the customer.

Model with One Price Break:-

Let D be the annual consumption (Demand)
C_1 is the price per unit (Basic price)
C_2 is the discounted price per unit.
C_0 is the ordering cost
C_h is the inventory carrying cost
Q_B is the priced break quantity.

With the following notations, suppose the following discount schedule is quoted by a supplier in which one price break (quantity discount) occurs at quantity 'b_1'.

QUNTITY	PRICE PER UNIT (RS)
$0 \leq Q_1 < b_1$	C_1

$b_1 \le Q_2$	C_2 $(< C_1)$

Procedure:-

Step 1:- Consider the lowest price (i.e. C_2) and determine Q_2^* using basic formula

$$Q_2^* = \sqrt{\frac{2\,D\,C_0}{C_h}}$$

If we find $Q_2^* > Q_B$ i.e. $Q_2^* \ge b_1$, then Q_2^* is the EOQ

$Q_2^* = Q^*$

$$TC^* (= TC_2^*) = DC_2 + \frac{D}{b_1} * C_0 + \frac{b_1}{2} * C_h$$

Step 2:- If $Q_2^* \ngeq b_1$, then calculate Q_1^* with price C_1, calculate also Total Cost at Q_1^*.

Compare TC (b_1) and TC (Q_1^*) .

If we get **TC (b_1) > TC (Q_1^*) , then EOQ is Q* = . Q_1^***

Otherwise, $Q^* = b_1$ is the required EOQ.

Problem 1:- Find the optimum order quantity for a produce for which the price breaks are as follows.

QUNTITY	PRICE PER UNIT (RS)
$0 \le Q_1 < 500$	10.00
$500 \le Q_2$	9.25

The monthly demand for the product is 200 units, the cost of storage is 2% of the unit cost and the cost of ordering is Rs 350.

Solution:-

Step 1:- Consider the lowest price (i.e. 9.25) and determine Q_2^* using basic formula

$$Q_2^* = \sqrt{\frac{2\,D\,C_o}{C_h}} = \sqrt{\frac{2*200*350}{9.25*0.02}} = 870 \text{ units.}$$

Now, $Q_2^* = 870$ units and $b_1 = 500$ units

$Q_2^* \geq b_1$, then Q_2^* is the EOQ

$Q_2^* = Q^*$

$Q_2^* = Q^* = 870$ units

Problem 2:- Find the optimum order quantity for a produce for which the price breaks are as follows.

QUNTITY	PRICE PER UNIT (RS)
$0 \leq Q_1 < 2000$	10.00
$2000 \leq Q_2$	9.25

The annual demand for the product is 10,500 units, the cost of storage is 30% of the unit cost and the cost of ordering is Rs 40.

<u>Solution:-</u>

Step 1:- Consider the lowest price (i.e. 9.25) and determine Q_2^* using basic formula

$$Q_2^* = \sqrt{\frac{2\,D\,C_o}{C_h}} = \sqrt{\frac{2*10500*40}{9.25*0.3}} = 543 \text{ units.}$$

Now, $Q_2^* = 543$ units and $b_1 = 2000$ units

$Q_2^* \ngeqq b_1$

Step 2:- If $Q_2^* \ngeqq b_1$, then calculate Q_1^* with price C_1, calculate also Total Cost at Q_1^*.

$$Q_1^* = \sqrt{\frac{2\,D\,C_o}{C_h}} = \sqrt{\frac{2*10500*40}{10*0.3}} = 529 \text{ units.}$$

$TC(Q_1^*)$ = TC (529 units) = $DC_1 + \frac{D}{Q_1} * C_0 + \frac{Q_1}{2} * C_h$

$= 10500 * 10 + \frac{10500}{529} * 40 + \frac{529}{2} * 10 * 0.30$

= 1,06,587.45/-

$TC(b_1) = DC_2 + \frac{D}{b_1} * C_0 + \frac{b_1}{2} * C_h = 10500 * 9.5 + \frac{10500}{2000} * 40 + \frac{2000}{2} * 9.25 * 0.30$

= 1,02,810/-

Since, **$TC(b_1) < TC(Q_1^*)$,** $Q^* = b_1$ is the required EOQ

Therefore, the optimum order quantity $Q^* = b_1 = 2000$ units.

***Model with TWO Price Break**:-*

Let D be the annual consumption (Demand)
C_1 is the price per unit (Basic price)
C_2 ($C_2 < C_1$) is the discounted price per unit.
C_3 ($C_3 < C_2$) is the discounted price per unit
C_0 is the ordering cost
C_h is the inventory carrying cost
Q_B is the priced break quantity.

With the following notations, suppose the following discount schedule is quoted by a supplier in which one price break (quantity discount) occurs at quantity 'b_1' and second price break occurs at quantity 'b_2'

QUNTITY	PRICE PER UNIT (RS)
$0 \leq Q_1 < b_1$	C_1
$b_1 \leq Q_2 < b_2$	C_2 ($< C_1$)
$b_2 \leq Q_3$	C_3 ($< C_2$)

Procedure:-

Step 1:- Consider the lowest price (i.e. C_3) and determine Q_3^* using basic formula

$$Q_3^* = \sqrt{\frac{2 D C_0}{C_h}}$$

If we find $Q_3^* \geq b_2$ then Q_3^* is the EOQ

$$Q_3^* = Q^* \text{ and Calculate TC } (Q_3^*)$$

If we find $Q_3^* < b_2$ then go to STEP 2

Step 2:- Calculate Q_2^* based on price C_2

- Compare Q_2^* with b_1
- If $b_1 \leq Q_2^* < b_2$, calculate TC (Q_2^*) and TC (b_2)
- If TC (b_2) $\leq$ TC (Q_2^*), EOQ = $b_2 = Q_2^*$
- If $Q_2^* < b_1$ as well as b_2, then go to STEP 3

  Step 3:- Calculate Q_1^* based on price C_1

  - Calculate TC (b_1), TC (b_2), TC (Q_1^*)
  - Compare among the above three
  - The quantity with lowest cost naturally be the required EOQ.

Problem 3:- Find the optimum order quantity for a produce for which the price breaks are as follows.

QUNTITY	PRICE PER UNIT (RS)
$0 \leq Q_1 < 100$ (b_1)	20.00 (C_1)
100 (b_1) $\leq Q_2 <$ 200 (b_2)	18.00 (C_2)
200 (b_2) $\leq Q_3$	16.00 (C_3)

The monthly demand for the product is 400 units, the cost of storage is 20% of the unit cost and the cost of ordering is Rs 25.

Solution:-

Step 1:- Consider the lowest price (i.e. 16.00) and determine Q_3^* using basic formula

$$Q_3^* = \sqrt{\frac{2\,D\,C_o}{C_h}} = \sqrt{\frac{2*400*25}{16*0.20}} = 79 \text{ units.}$$

we find Q_3^* (= 79 units) < b_2 (= 200) then go to STEP 2

Step 2:- Calculate Q_2^* based on price C_2

$$Q_2^* = \sqrt{\frac{2\,D\,C_o}{C_h}} = \sqrt{\frac{2*400*25}{18*0.20}} = 75 \text{ units}$$

- If Q_2^* (= 75 units) < b_1 (= 100) as well as b_2 (= 200) , then go to STEP 3

Step 3:- Calculate Q_1^* based on price C_1

$$Q_1^* = \sqrt{\frac{2\,D\,C_o}{C_h}} = 71 \text{ units}$$

- Calculate TC (b_1) , TC (b_2) , TC (Q_1^*)

$$TC(b_1) = DC_2 + \frac{D}{b_1} * C_0 + \frac{b_1}{2} * C_h = 400 * 18 + \frac{400}{100} * 25 + \frac{100}{2} * 18 * 0.20 = 7480/-$$

$$TC(b_2) = DC_3 + \frac{D}{b_2} * C_0 + \frac{b_2}{2} * C_h = 400 * 16 + \frac{400}{200} * 25 + \frac{200}{2} * 16 * 0.20 = 6770/-$$

$$TC(Q_1^*) = DC_1 + \frac{D}{Q_1*} * C_0 + \frac{Q_1*}{2} * C_h = 400 * 20 + \frac{400}{71} * 25 + \frac{71}{2} * 20 * 0.20 = 8283/-$$

Since , TC (b_2) < TC (b_1) < TC (Q_1^*)

The optimum order quantity is given by

$Q^* = b_2 = 200$ units.

Problem 4:- A shop keeper has a uniform demand of an item at the rate of 50 items per month. He buys from a supplier at a cost of Rs 6 per item and the cost of ordering is Rs 10 for each order. If the stock holding costs are 20% per year of stock value, how frequently should he replenish his stocks?

Now, suppose the supplier offers a 5% discount on orders between 200 and 999 items and a 10% discount on orders exceeding or equal to 1000 items. Can the shop keeper reduce his costs by taking advantage of either of these discounts?

Solution:-

Given, D = 50*12 = 600 items per year, C_0 = Rs 10 per order,

C_p = Rs 6 per item, $C_h = C_p * I = 6*0.20 = 1.2$

$$Q^* = \sqrt{\frac{2 D C_0}{C_h}} = \sqrt{\frac{2*600*10}{6*0.20}} = 100 \text{ items}$$

$$T^* = \frac{Q^*}{D} = \frac{100}{600} = \frac{1}{6} \text{ year} = 2 \text{ months}$$

$$TC(Q^*) = DC_p + \frac{D}{Q*} * C_0 + \frac{Q*}{2} * C_h = 600*6 + \frac{600}{100}*10 + \frac{100}{2}*6*0.20 = 3720/-$$

In the case of discounts we have the following formulation

QUNTITY	PRICE PER UNIT (RS)
$0 \leq Q_1 < 200 \; (b_1)$	6.00 ($= C_1$)
$200 \; (b_1) \leq Q_2 < 1000 \; (b_2)$	5.70(5% discount) ($= C_2$)
$1000 \; (b_2) \leq Q_3$	5.40 (10% discount) ($= C_3$)

Step 1:- Consider the lowest price (i.e. 5.40) and determine Q_3^* using basic formula

$$Q_3^* = \sqrt{\frac{2 D C_0}{C_h}} = \sqrt{\frac{2*600*10}{5.40*0.20}} = 105 \text{ units}$$

we find Q_3^* (= 105 units) < b_2 (= 1000) <u>then go to STEP 2</u>

Step 2:- Calculate Q_2^* based on price C_2

$$Q_2^* = \sqrt{\frac{2\,D\,C_o}{C_h}} = \sqrt{\frac{2*600*10}{5.70*0.20}} = 103 \text{ units}$$

- If Q_2^* (= 103 units) < b_1 (= 200) as well as b_2 (= 1000) , <u>then go to STEP 3</u>

Step 3:- Calculate Q_1^* based on price C_1

$$Q_1^* = \sqrt{\frac{2\,D\,C_o}{C_h}} = \sqrt{\frac{2*600*10}{6*0.20}} = 100 \text{ units}$$

- Calculate TC (b_1) , TC (b_2) , TC (Q_1^*)

$$TC(b_1) = DC_2 + \frac{D}{b_1} * C_0 + \frac{b_1}{2} * C_h = 600 * 5.70 + \frac{600}{200} * 10 + \frac{200}{2} * 5.70 * 0.20 = 3564/-$$

$$TC(b_2) = DC_3 + \frac{D}{b_2} * C_0 + \frac{b_2}{2} * C_h = 600 * 5.40 + \frac{600}{1000} * 10 + \frac{1000}{2} * 5.40 * 0.20 = 3786/-$$

$$TC(Q_1^*) = DC_1 + \frac{D}{Q_1^*} * C_0 + \frac{Q_1^*}{2} * C_h = 600 * 6 + \frac{600}{100} * 10 + \frac{100}{2} * 6 * 0.20 = 3720/-$$

Since , TC (b_1) < TC (Q_1^*) < TC (b_2)

The optimum order quantity is given by

<u>**Q^* = b_1 = 200 units.**</u>

The shop keeper should accept the offer at 5% discount only as by doing this he is able to save Rs 3720 – 3564 = Rs 156 during the year.

For further practice, please refer

1. Operations Research- Theory and Applications – J.K.Sharma

2. Operations Research- Problems and Solutions – V.K.Kapoor

QUEUING THEORY:-

The queuing theory is known as Random System Theory which has the solutions for statistical interference and problem of behavior and optimization in queuing system. Indeed, queuing theory has many applications in human endeavors, some of which include: telephony; manufacturing; inventories; dams; supermarkets; computer and information communication systems and networks; call centers; hospitals, banking, etc.

Undoubtedly, there are numerous factors that affect a customer's perception of the waiting experience, some of which include: physical, psychological and emotional. If there were to be no queue at all, it would create the impression that the value of the attraction is to some extent diminished. However, one may observe that attractions with short queues tend to attract less public. So, in principle, it is important not to aim at eliminating queues, but instead concentrate on giving people an option to join the queue, or skip part of the queue and spend the time somewhere else.

A flow of customers from finite or infinite population towards the service facility forms a **queue (waiting line)** an account of lack of capability to serve them all at a time. In the absence of a perfect balance between the service facilities and the customers, **waiting time** is required either for the service facilities or for the customers' arrival. In general, the **queuing system** consists of one or more queues and one or more servers and operates under a set of procedures. Depending upon the server status, the incoming customer either waits at the queue or gets the turn to be served. If the server is free at the time of arrival of a customer, the customer can directly enter into the counter for getting service and then leave the system. In this process, over a period of time, the system may experience "Customer waiting" and /or "Server idle time"..

Queuing System:

A queuing system can be completely described by

(1) the input (arrival pattern)
(2) the service mechanism (service pattern)
(3) The queue discipline and
(4) Customer's behaviour

The input (arrival pattern)

The input described the way in which the customers arrive and join the system. Generally, customers arrive in a more or less random manner which is not possible for prediction. Thus the arrival pattern can be described in terms of probabilities and consequently the probability distribution for **inter-arrival** times (the time between two successive arrivals) must be defined. We deal with those Queuing system in which the customers arrive in poisson process. The mean arrival rate is denoted by λ.

The Service Mechanism:-

This means the arrangement of service facility to serve customers. If there is infinite number of servers, then all the customers are served instantaneously or arrival and there will be no queue. If the number of servers is finite then the customers are served according to a specific order with service time a constant or a random variable. Distribution of service time follows '**Exponential distribution'** defined by

$$f(t) = \lambda e^{-\lambda t}, \; t > 0$$

The mean Service rate is $E(t) = 1/\lambda$

Queuing Discipline:-

It is a rule according to which the customers are selected for service when a queue has been formed. The most common disciplines are

1. First come first served – (FCFS)
2. First in first out – (FIFO)
3. Last in first out – (LIFO)
4. Selection for service in random order (SIRO)

Customer's behaviour

1. Generally, it is assumed that the customers arrive into the system one by one. But in some cases, customers may arrive in groups. Such arrival is called **Bulk arrival.**

2. If there is more than one queue, the customers from one queue may be tempted to join another queue because of its smaller size. This behaviour of customers is known as **jockeying.**

3. If the queue length appears very large to a customer, he/she may not join the queue. This property is known as **Balking** of customers.

4. Sometimes, a customer who is already in a queue will leave the queue in anticipation of longer waiting line. This kind of departure is known as **reneging.**

The dynamics of queues have been analyzed by using steady-state mathematics. Essentially, it is purely a mathematical approach that is employed in the waiting line analysis. While various models constitute several queuing systems such queuing processes are described by using the **Kendall-Lee (1953)** notation which uses mnemonic characters that specify the queuing system:

A/B/C/D/E/F

- A: Specifies the nature of the arrival process.
- B: Specifies the nature of the service times.
- C: Specifies the number of parallel servers
- D: Specifies the queue discipline.

– E: Specifies the maximum number of entities in the system.

– F: Specifies the size of the population from which entities are drawn.

Characteristics of a queuing process

The queuing theory considers mainly six general characteristics of any queuing processes:

i) **Arrival pattern of customers**: inter-arrival times most commonly fall into one of the following distribution patterns: A Poisson distribution, a Deterministic distribution, or a General distribution. However, inter-arrival times are most often assumed to be independent and memory less, which is the attributes of a Poisson distribution.

ii) **Service pattern:** the service time distribution can be constant, exponential, hyper exponential, hypo-exponential or general. The service time is independent of the inter-arrival time.

iii) **Number of servers:** the queuing calculations change depends on whether there is a single server or multiple servers for the queue. A single server queue has one server for the queue. This is the situation normally found in a grocery store where there is a line for each cashier.

iv) **Queue Lengths:** the queue in a system can be modeled as having infinite or finite queue length.

v) **System capacity**: the maximum number of customers in a system can be from 1 up to infinity. This includes the customers waiting in the queue.

vi) **Queuing discipline**: there are several possibilities in terms of the sequence of customers to be served.

- FCFS**:** First Come, First Served. This is the most commonly used discipline applied in the real-world situations, such as check-in counters at the airport.
- LCFS**:** Last Come, First Served**.** This illustrates a reverse order service given to customer versus their arrival.
- SIRO**:** Service in Random Order.
- PD**:** Priority Discipline. Under this discipline, customers will be classified into categories of different priorities.

List of Variables

The list of variables used in queuing models is give below:

n - No of customers in the system

C - No of servers in the system

P_n (t) – Probability of having n customers in the system at time t.

P_n - Steady state probability of having customers in the system

P_0 - Probability of having zero customers in the system

L_q - Average number of customers waiting in the queue.

L_s - Average number of customers waiting in the system (in the queue and in the service counters)

W_q - Average waiting time of customers in the queue.

W_s - Average waiting time of customers in the system (in the queue and in the service counters)

λ - Arrival rate of customers

μ - Service rate of server

ρ - Utilization factor of the server

M - Poisson distribution

N - Maximum numbers of customers permitted in the system. Also, it denotes the size of the calling source of the customers.

GD - General discipline for service. This may be first in first – serve (FIFS), last-in-first serve (LIFS) random order (Ro) etc.

Classification of Queuing models

Generally, queuing models can be classified into six categories using Kendall's notation with Five parameters to define a model. The parameters of this notation are

a- Arrival rate distribution i.e. probability law for the arrival /inter – arrival time.
b- Service rate distribution, i.e. probability law according to which the customers are being served.
c - Number of Servers (i.e. number of service stations)

d - Service discipline
e - Maximum number of customers permitted in the system.

A queuing model with the above parameters is written as (a/b/c : d/e)

Model 1 : (M/M/1) : (∞/ FCFS) Model

In this model

(i) the arrival rate (λ) follows Poisson (M) distribution.
(ii) Service rate (μ), service times follow exponential distribution (M)
(iii) Number of servers is 1
(iv) Service discipline is general disciple (i.e. FCFS)
(v) Maximum number of customers permitted in the system is infinite (∞)

List of Equations (without proof) under Model I

1. Utilistion parameter:

$$\rho = \frac{\lambda}{\mu}$$

2. Probability that the system is idle:

$$P_0 = 1 - \frac{\lambda}{\mu}$$

3. Expected number of customers in the system:

$$L_s = \frac{\lambda}{\mu - \lambda}$$

4. Expected number of customers waiting in the queue:

$$L_q = \frac{\lambda^2}{\mu(\mu - \lambda)}$$

5. Expected length of non empty queue :

$$L_q' = \frac{\mu}{\mu - \lambda}$$

6. Expected waiting time in queue :

$$W_q = \frac{\lambda}{\mu\,(\mu - \lambda)}$$

7. Expected time a customer spends in the system :

$$W_s = \frac{1}{(\mu - \lambda)}$$

8. Probability that there will be ‘k’ or more units in the system :

$$P(n \geq k) = \left(\frac{\lambda}{\mu}\right)^k$$

9. Probability that a customer shall wait for more than ‘t’ times in the queue

$$= \rho * e^{-t/w_s}$$

Problem 1:-

The arrival rate of customers at a banking counter follows a poisson distibution with a mean of 30 per hours. The service rate of the counter clerk also follows poisson distribution with mean of 45 per hour.

a) What is the probability of having zero customer in the system ?
b) What is the probability of having 8 customer in the system ?
c) Find Ls, Lq, Ws and Wq

Solution

Given arrival rate follows poisson distribution with mean =30

λ= 30 per hour

Given service rate follows poisson distribution with mean = 45

μ = 45 per hour

Utilization factor $\rho = \frac{\lambda}{\mu}$

= 30/45

= 2/3

= 0.67

a) The probability of having zero customer in the system $P_0 = 1\ 1 - \frac{\lambda}{\mu}$

= 1- 0.67 = 0.33

b) The probability of having 8 customers in the system =

$P(n = 8) = \left(\frac{2}{3}\right)^8 (1 - \frac{2}{3})$ = 0.0130

c) $L_s = \frac{\lambda}{\mu - \lambda}$ = 2 customers

$L_q = \frac{\lambda^2}{\mu(\mu - \lambda)}$ = 1 customer

$W_s = \frac{1}{(\mu - \lambda)}$ = 0.0666 hour

$W_q = \frac{\lambda}{\mu(\mu - \lambda)}$ = 0.4467 hour

Problem 2:-

XYZ Tailoring house has one tailor specialized in men's shirts. The number of customers requiring stitching of shirts appears to follow Poisson distribution with mean arrival rate of 12 per hour. Customers are attended to tailor on a first cum first served basis, and they are willing to wait for service if there be queue. The time tailor takes to attend a customer is exponentially distributed with a mean of 4 minutes. Required

i) The utilization parameter

ii) The probability that the queuing system is idle.

iii) The average time the tailor is free on 8 hour working days.

iv) What is the probability that there shall be 5 customers in the shop?

v) What is the number of customers in the shop?

vi) What is the number of customers waiting for tailor's services?

vii) What is the expected length of non empty queue?

viii) How much time a customer should expect to spend in the queue?

ix) How much time should a customer expect to spend in the shop?

x) What is the probability that a customer shall spend more than 10 minutes.

Solution

Given, λ= 12 customers per hour, μ= 15 customers per hour

i) Utilization parameter:

$$\rho = \frac{\lambda}{\mu} = 0.8$$

ii) The probability that the queuing system is idle:

$$P_0 = 1 - \frac{\lambda}{\mu} = 0.2$$

iii) The average time the tailor is free on 8 hour working days

$$= P_0 * \text{No of hours} = 0.2*8 = 1.6 \text{ hours}$$

iv) The probability that there shall be 5 customers in the shop

$$P_k = \left(1 - \frac{\lambda}{\mu}\right) * \left(\frac{\lambda}{\mu}\right)^k = (1 - 0.8) * (0.8)^5 = 0.0655$$

v) Expected number of customers in the shop

$$L_s = \frac{\lambda}{\mu - \lambda} = 4 \text{ customers}$$

vi) Expected number of customers waiting for tailor's services

$$L_q = \frac{\lambda^2}{\mu(\mu - \lambda)} = 3.2 \text{ costomers}$$

vii) The expected length of non empty queue

$$L_q' = \frac{\mu}{\mu - \lambda} = 5 \text{ customers}$$

viii) Expected time a customer should expect to spend in the queue

$$W_q = \frac{\lambda}{\mu(\mu - \lambda)} = 16 \text{ minutes}$$

ix) Expected time a customer should expect to spend in the shop

$$W_s = \frac{1}{(\mu - \lambda)} = 20 \text{ minutes}$$

x) Probability that a customer shall spend more than 10 minutes

$$\rho * e^{-t/W_s} = 0.8 * e^{-10/20} = 0.49$$

For further practice, please refer

1. Operations Research- Theory and Applications – J.K.Sharma

2. Operations Research- Problems and Solutions – V.K.Kapoor

University of Calcutta

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CC-203 --- Operations Research- Module –II

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SEQUENCING PROBLEMS

In this chapter, we shall try to determine an appropriate order(sequence) for a series of jobs to be done on a finite number of service facilities in some pre-assigned order, so as to optimize the total cost(time) involved.

Sequencing technique deals with the problem of preparing optimal time table for jobs, equipments, people, materials, facilities and all other resources that are needed to support the production schedule. The objective is the minimization of the total elapsed time between the completion of first and last job in a particular order.

It gives us an idea of the order in which things happen or come in event. Suppose, there are n jobs(1,2,3……n), each of which has to be processed one at a time at m machines(*A*,*B*,*C*….).The order of processing each job through each machine is given. The problem is to find a sequence among $(n!)^m$ number of all possible sequences for processing the jobs so that the total elapsed time for all the jobs will be minimum.

Terminology and Notations:

The following are the terminologies and notations used in this sequencing problem:

Number of machines: It means the service facilities through which a job must pass before it is completed.

Processing order: It refers to the order (sequence) in which various machines are required for completing the job.

Processing Time: It means the time required by each job to complete a prescribed procedure on each machine.

Idle time on a machine:. This is the time for which a machine remains idle during the total elapsed time. During the time, the machine awaits completion of manual work. The notation x_{ij} is used to denote the idle time of a machine j between the end of the $(i-1)$ ***th*** job and start of the i ***th*** job.

Total elapsed time: This is the time interval between starting the first job and completing the last job, which also includes the idle time, if it occurs.

No passing rule: It means, passing is not allowed. i.e maintaining the same order of jobs over each machine. If each of ***N*** jobs is to be processed through 2 machines M_1 and M_2 in the order

$M_1 M_2$, then this rule will mean that each job will go to machine M_1 first and then to M_2 . If a job is finished on M_1, it goes directly to machine M_2 , if it is free, otherwise it starts a waiting line or joins the end of the waiting line, if one already exists. Jobs that form a waiting line are processed on machine M_2 when it becomes free.

Principal Assumptions:

i) The processing time on different machines are exactly known and are independent of the order of the jobs in which they are to be processed.
ii) No machine can process more than one operation at a time.
iii) Each operation once started must be performed till completion.
iv) Each operation must be completed before starting any other operation.
v) Time intervals for processing are independent of the order in which operation are performed.
vi) There is only one machine of each type.
vii) A job is processed as soon as possible, subject to the ordering requirement.
viii) All jobs are known and are ready for processing, before the period under consideration begins.
ix) The time required to transfer the jobs between machines is negligible.

TYPE-1: PROBLEMS WITH *n* JOBS THROUGH TWO MACHINES

The algorithm, which is used to optimize the total elapsed time for processing n jobs through two machines is called '**Johnson's algorithm'** and has the following steps: Consider n jobs (1,2,3….n) processing on two machines A and B in the order AB. The processing periods (time) are $A_1, A_2, A_3, \dots\dots\dots . A_n$ and $B_1, B_2, B_3, \dots\dots\dots . B_n$ as given in the following table.

Machines/jobs	1	2	3	…….	n
A	A_1	A_2	A_3	……	A_n
B	B_1	B_2	B_3	…….	B_n

The problem is to sequence the jobs so as to minimize the total elapsed time.

The solution procedure adopted by Johnson is given below.

Step-1: select the least processing time occurring in the list $A_1, A_2, A_3, \dots\dots\dots . A_n$ and $B_1, B_2, B_3, \dots\dots\dots . B_n$ .Let this minimum processing time occurs for a job ***K***.

Step-2: If the shortest processing is for machine A, process the ***K th*** job **<u>first</u>** and place it **<u>in the beginning of the sequence</u>.** If it is for machine B, process the ***K th*** job **<u>last</u>** and place it **<u>at the end of the sequence.</u>**

Step-3:When there is a ***tie*** in selecting the minimum processing time , then there may be **three** solutions:

(i) If the equal minimum values occur only for machine A, select the job with **larger processing time in B** to be placed **first in the job sequence**.
(ii) If the equal minimum values occur only for machine B, select the job with **larger processing time in A** to be placed **last in the job sequence**.
(iii)If there are equal minimum values, one for each machine, then place the job in **machine A first and the one in machine B last**.

Step-4:

Delete the jobs already sequenced, If all the jobs have been sequenced, go to the next step.

Step-5:

In this step, determine the overall or total elapsed time and also the idle time on machine and B as follows:

Total elapsed time=The time between starting the first job in the optimal sequence on machine A and completing the last job in the optimal sequence on machine B.

Idle time on A=(Time when the last job in the optimal sequence is completed on machine B)-(Time when the last job in the optimal sequence is completed on machine A)

Idle time on B

= (When the first job in the optimal sequence starts on machine B)+$\sum_{K=2}^{n}[\text{time k th job starts on machine B} - \text{time}(K-1)\text{th job finished on machine B}]$

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Practical Problem:-1

There are five jobs, each of which must go through the machines A and B in the order AB. Processing times are given below.

Jobs	1	2	3	4	5
Machine A	5	1	9	3	10
Machine B	2	6	7	8	4

Determine a sequence for the five jobs that will minimize the total elapsed time.

Solution:

The shortest processing time in the given problem is 1 on machine A. So, perform job 2 in the beginning, as shown below.

2				

The reduced list of processing time becomes

Jobs	1	3	4	5
Machine A	5	9	3	10
Machine B	2	7	8	4

Again the shortest processing time in the reduced list is 2 for job 1 on machine B. So place job 1 as the last.

2				1

Continuing in the same manner, the next reduced list is obtained as :

Jobs	3	4	5
Machine A	9	3	10
Machine B	7	8	4

Leading to the sequence

2	4			1

and reduced list is as follows:

Jobs	3	5
Machine A	9	10
Machine B	7	4

It gives rise to the sequence:

2	4		5	1

Finally the optimal sequence n is obtained as:

2	4	3	5	1

Therefore, the flow of jobs through machines A and B using the optimal sequence is:

$$2 \rightarrow 4 \rightarrow 3 \rightarrow 5 \rightarrow 1$$

job	**Machine A**		**Machine B**		**Idle time**	
	In	**Out**	**In**	**Out**	**A**	**B**
2	0	1	1	7	0	1
4	1	4	7	15	0	0
3	4	13	15	22	0	0
5	13	23	23	27	0	1
1	23	28	28	**(30)**	30-28	1
					=2	3

From the above table, we find that the total elapsed time is 30 hours and the idle time on machine A is 2 hours and on machine B is 3 hours.

Practical Problem:-2

Find the sequence that minimizes the total elapse time (in hours) required to complete the following tasks on two machines.

Task	A	B	C	D	E	F	G	H	*I*
Machine I	2	5	4	9	6	8	7	5	4
Machine II	6	8	7	4	3	9	3	8	11

Solution:

The shortest processing time is 2 hours on machine-I for job A. Hence, process this job first.

A								

Deleting this job, we get the reduced list of processing time.

Task	B	C	D	E	F	G	H	*I*
Machine I	5	4	9	6	8	7	5	4
Machine II	8	7	4	3	9	3	8	11

The next minimum processing time is same for jobs E and G on machine II. The corresponding processing time on machine I for this job is 6 and 7. The longest processing time is 7 hours. So sequence the job G at the end and E next to it.

A							E	G

Deleting the jobs that sequenced, the reduced processing list is:

Task	B	C	D	F	H	*I*
Machine I	5	4	9	8	5	4
Machine II	8	7	4	9	8	11

The minimum processing time is 4 hours for job C, *I* and D. For job C and *I* , it is on machine I and for job D , it is on machine II. There is a tie in sequencing jobs C and *I*. In order to break this, we consider the corresponding time on machine II, the longest time is 11(eleven) hours. Hence, sequence job *I* in the beginning followed by job C. For job D, as it is on machine II, sequence it last.

A	*I*	C				D	E	G

Deleting the jobs that are sequenced , the reduced processing list is:

Task	B	F	H
Machine I	5	8	5
Machine II	8	9	8

The next minimum processing time is 5 hours on machine I for job B and H, which is again a tie. In order to break this, we consider the corresponding longest time on other machine(II) and sequence the job B or H first.

Finally, job F is sequenced.

The optimal sequence for this job is:

A	*I*	C	B	H	F	D	E	G

The total elapsed time and idle time for both the machines are calculated from the following table:

Task	**Machine I**		**Machine II**		**Idle time**	
	In	**Out**	**In**	**Out**	$\boldsymbol{M_1}$	$\boldsymbol{M_2}$
A	0	2	2	8	0	2
I	2	6	8	19	0	0
C	6	10	19	26	0	0
B	10	15	26	34	0	0
H	15	20	34	42	0	0
F	20	28	42	51	0	0
D	28	37	51	55	0	0
E	37	43	55	58	0	0

G	43	50	58	61	61-50	0
					11 Hours	2 Hours

Total elapsed time=61 Hours

Idle time for machine I=11 Hours; Idle time for machine II=2 Hours.

TYPE-1: PROBLEMS WITH *n* JOBS THROUGH THREE MACHINES-A,B,C

Consider n jobs(1,2,3…..n) processing on three machines *A*,*B*,*C* in the order *ABC.* The optimal sequence can be obtained by converting the problem into a two-machine problem. From this, we get the optimum sequence using **Johnson's algorithm**.

The following steps are used to convert the given problem into a two-machine problem.

Step-1: Find the minimum processing time for the jobs on the first and last machine and the maximum processing time for the second machine,i.e

find $$Min_i(A_i, C_i), \quad i=1,2,3\ldots\ldots n$$

and

$$Max_i(B_i)$$

Step-2: Check the following inequality

$$Min_i A_i \geq Max_i B_i$$

or

$$Min_i C_i \geq Max_i B_i$$

Step-3: If none of the inequalities in step 2 are satisfied, this method can not be applied.

Step-4: If at least one of the inequalities in step 2 is satisfied, we define two machines G and H, such that the processing time on G and H are given by,

$$G_i = A_i + B_i \quad , i=1,2,3\ldots.n$$

$$H_i = B_i + C_i \quad , i=1,2,3\ldots.n$$

Step-5: For the converted machine G and H, we obtain the optimum sequence using two machine algorithm.

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Practical problem-1:

A machine operator has to perform three operations, turning , threading and knurling, on a number of different jobs. The time required to perform these operations(in minutes) on each job is known. Determine the order in which the jobs should be processed in order to minimize the total time required to turn out all the jobs. Also find the minimum elapsed time.

Job	1	2	3	4	5	6
turning	3	12	5	2	9	11
threading	8	6	4	6	3	1
knurling	13	14	9	12	8	13

Solution:

Let us consider three machines as A,B and C

A=Turning , B=Threading, C= Knurling

Step-1:

$Min_i(A_i, C_i)$=(2,8)

$Max_i(B_i)$=8

Step-2:

$Min_i A_i = 2 \ngeq Max_i B_i$=8

$Min_i C_i$ =8$\geq Max_i B_i$=8 is satisfied.

we define two machines G and H

such that , $G_i = A_i + B_i$

$H_i = B_i + C_i$

Job	1	2	3	4	5	6
G	11	18	9	8	12	12
H	21	20	13	18	11	14

We adopt Johnson's algorithm steps to get the optimum sequence.

4	3	1	6	2	5

In order to find the total elapsed time and idle time for machine A,B and C,

Job	Machine A		Machine B		Machine C		Idle time		
	In	***Out***	***In***	***Out***	***In***	***Out***	A	B	C
4	0	2	2	8	8	20	-	2	8
3	2	7	8	12	20	29	-	-	-
1	7	10	12	20	29	42	-	-	-
6	10	21	21	22	42	55	-	1	-
2	21	33	33	39	55	69	-	11	-
5	33	42	42	45	69	77		3	-
							77-42	(77-45)+17	-
							35	49	8

Total elapsed time=77 minutes

Idle time for machine A=35 minutes; Idle time for machine B=49 minutes;

Idle time for machine C=8 minutes.

Study material prepared by Dr S.P.Ray

For further practice, please refer

1. Operations Research- Theory and Applications – J.K.Sharma

2. Operations Research- Problems and Solutions – V.K.Kapoor